Clones between Group $(\mathbb{Z}_8, +)$ and Ring $(\mathbb{Z}_8, +, \cdot)$

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SSAOS 2025,
Českovice, Czech Republic

8.9.2025

Funded by the European Union
NextGenerationEU

Funded by the EU NextGenerationEU through the Recovery and Resilience Plan for Slovakia under the project No. 09I03-03-V05-00008.

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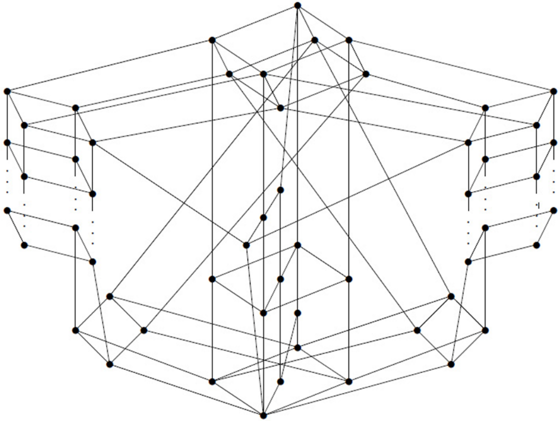
If $|A| = 2$, then the clone lattice on A consists of countably many clones (Post's Lattice).



Lattice of Clones on a Set

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An intersection of any system of clones on the set A forms a clone on A.

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All clones on a set A form a complete lattice.

- *If $|A| = 1$, then the clone lattice on A consists of one clone.*
- *If $|A| = 2$, then the clone lattice on A consists of countably many clones (Post's Lattice).*
- *If $|A| > 2$, then the clone lattice on A consists of uncountable clones.*

Interval $\mathcal{J}_n$ in the Clone Lattice

$\mathcal{J}_n = \langle P(\mathbb{Z}_n, +), P(\mathbb{Z}_n, +, \cdot) \rangle$

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The elements of $P(\mathbb{Z}_n, +)$ are all linear functions in the following form

$$p(x_1, \ldots, x_n) = a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n,$$

where $a_0 \in \mathbb{Z}_n$, $a_1, \ldots, a_n \in \mathbb{Z}$.

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$$q(\mathbf{x}) = \sum_{\alpha} a_\alpha \mathbf{x}^\alpha,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$*, the sum consists of finitely many tuples* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *of natural numbers, coefficients* a_α *belong to the set* $\mathbb{Z}_n$ *and* $\mathbf{x}^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

What Is Already Known about the Interval $\mathcal{J}_n = \langle P(\mathbb{Z}_n,+), P(\mathbb{Z}_n,+,\cdot)\rangle$?

Solved cases:

- $n =$ prime number p (Rosenberg, 1970)

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Lemma

Let n, m be prime numbers. If $(m, n) = 1$, then

$$\mathcal{J}_{mn} \cong \mathcal{J}_m \times \mathcal{J}_n.$$

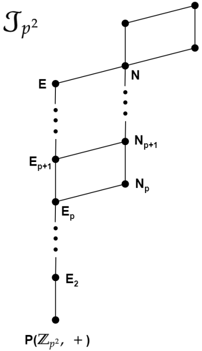
Corollary

It suffices to investigate the case $n = p^k$, where p is a prime number.

What Is Already Known about the Interval $\mathcal{J}_{p^k} = \langle P(\mathbb{Z}_{p^k}, +), P(\mathbb{Z}_{p^k}, +, \cdot)\rangle$?

Solved cases:

- $n = p^2$, where p is prime number (Krokhin et al. 1997; Idziak and Bulatov, 2003)



Open problem

What is the structure of the clone lattice of the interval $\mathcal{J}_{p^3}$?

Our result:
A complete characterization of the interval $\langle P(\mathbb{Z}_8, +), M_1 \rangle \subseteq \mathcal{J}_8$ in the clone lattice, where

$$\mathcal{J}_8 = \langle P(\mathbb{Z}_8, +), P(\mathbb{Z}_8, +, \cdot) \rangle, \quad (p = 2).$$

Used tools:

- 2^k-ary relations, where their elements are $\mathbf{x} = (x_A |\ A \in \mathcal{P}_k)$.
- (Möbius basis) We define $\mathbf{g}^A = (g_B^A\ | B \in \mathcal{P}_k) \in \mathbb{Z}_8^{P_k}$ for every $A \in \mathcal{P}_k$ as follows

$$g_B^A = \begin{cases} 1, & \text{if } A \subseteq B; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma

Every $\mathbf{x} \in \mathbb{Z}_8^{\mathcal{P}_k}$ can be expressed in the form

$$\mathbf{x} = \sum_{A \in \mathcal{P}_k} a_A \mathbf{g}^A,$$

where

$$a_A = (-1)^{|A|} \sum_{B \subseteq A} (-1)^{|B|} x_B$$

for every $A \in \mathcal{P}_k$. The expression of the element $\mathbf{x}$ in this form is uniquely determined.

Definition of the Clone M_1 via an Invariant Relation

The clone M_1 consists of all ring polynomials $\mathbb{Z}_8$ that preserve the relation Z.

Definition

The relation Z is 2^4-ary relation on $\mathbb{Z}_8$, that consists of all elements $\mathbf{u} = (u_A \mid A \in \mathcal{P}_4)$ satisfying:

(1) $a_2 \equiv 2a_1 \pmod 4$, $a_4 \equiv 2a_3 \pmod 4$;

(2) $a_A \equiv 0 \pmod 2$, *if* $|A| \geq 2$;

(3) $a_A \equiv 0 \pmod 4$, *if* $|A| \geq 2$, $A \cap \{2, 4\} \neq \emptyset$;

(4) $a_A = 0$, *if* $\{2, 4\} \subseteq A$.

Definition of the Clone M_1 by Its Generators

Definition

Let $f(x_1, \ldots, x_n)$ be a polynomial with variables $x_1, \ldots, x_n$. A polynomial $f(x_1, \ldots, x_n)$ is fully divisible if it is divisible by $x_1 x_2 \ldots x_n$.

Lemma (Bulatov)

Every clone on the interval $\mathcal{J}_{p^k}$ is generated by its fully divisible members.

Definition of the Clone M_1 by Its Generators

Lemma

Let be $n \geq 2$, then n-ary fully divisible polynomial on $\mathbb{Z}_8$ preserve the relation Z iff it can be expressed in the form

$$f = 2x_1 \dots x_n(\sum_{i=1}^{n} a_i x_i^2 + \sum_{i=1}^{n} b_i x_i + c),$$

where $a_i, b_i \in \{0, 1\}$ for all i a $c \in \{0, 1, 2, 3\}$.

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Lemma

Unary fully divisible polynomial on $\mathbb{Z}_8$ preserve Z iff it can be expressed in the form

$$f = ax^3 + bx^2 + cx,$$

where a, b are even.

What Are the Generators of Clones on the Interval $\langle P(\mathbb{Z}_8, +), M_1 \rangle \subseteq \mathcal{J}_8$?

Let us denote some n-ary $(n \geq 1)$ operations on $\mathbb{Z}_8$:

$$r_n = x_1 x_2 \ldots x_n;$$

$$t_n = \begin{cases} x_1 x_2 \ldots x_n (x_1 + \cdots + x_n), & \text{if } n \text{ is even;} \\ x_1 x_2 \ldots x_n (x_1 + \cdots + x_n + 1), & \text{if } n \text{ is odd;} \end{cases}$$

$$s_n = \begin{cases} x_1 x_2 \ldots x_n (x_1 + \cdots + x_n), & \text{if } n \text{ is even;} \\ x_1 x_2 \ldots x_n (x_1 + \cdots + x_n + 1), & \text{if } n \text{ is odd;} \end{cases}$$

$$u_n = x_1 x_2 \ldots x_n (x_1 + 1);$$

$$v_n = x_1 x_2 \ldots x_n (x_1 + x_2);$$

$$p_n = x_1^2 x_2 \ldots x_n;$$

$$q_n = x_1^3 x_2 \ldots x_n.$$

$C(f)$ = a clone generated by operation f, addition and constants.

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$C(f)$ = a clone generated by operation f, addition and constants.

Lemma

If $C(f) \subseteq M_1$, then $C(f)$ can be expressed in the form

$$C(2t_{n_1}) \vee C(2u_{n_2}) \vee C(2v_{n_3}) \vee C(2s_{n_4}) \vee C(2p_{n_5}) \vee C(2q_{n_6}) \vee C(2r_{n_7}) \vee C(4r_{n_8}).$$

Ordering of Clones on $\langle P(\mathbb{Z}_8, +), M_1\rangle$

Lemma

Let be $n \geq 1$*, then*

(i) $C(2p_n) \subseteq C(2q_n) \subseteq C(2p_{n+1})$;

(ii) $C(2p_n) \subseteq C(2r_{n+1}) \subseteq C(2q_{n+1}) \cap C(2r_{n+2})$;

(iii) $C(2s_n) \cup C(2u_n) \subseteq C(2r_{n+1})$;

(iv) $C(2v_{n+1}) \subseteq C(2q_n)$;

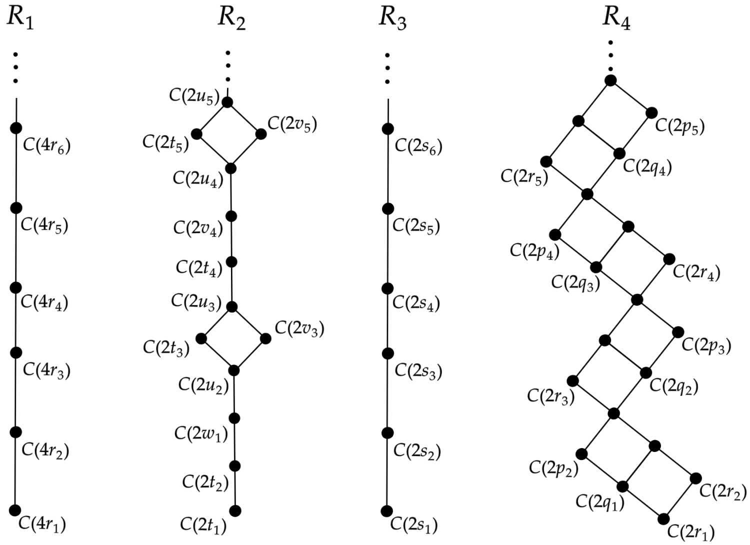
(v) $C(2q_1) \subseteq C(2u_2)$;

(vi) *if* n *is odd, then* $C(2s_n) \subseteq C(2p_n)$;

(vii) *if* n *is even, then* $C(2s_n) \subseteq C(2q_n)$;

(viii) $C(2r_n) \subseteq C(2p_{n+1}) \subseteq C(2r_{n+2})$.

$\vdots$


R_1
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$C(4r_2)$
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R_2
$C(2u_5)$
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$C(2t_5)$
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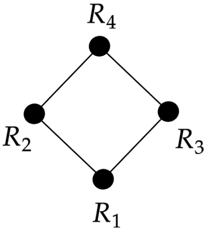
Ordering of Clones on $\langle P(\mathbb{Z}_8, +), M_1 \rangle$



Definition of the Clones via Invariant Relations

We define the following conditions depending on k:

(C1) $a_{2m} \equiv 2a_{2m-1} \pmod 4$ for $m = 1, \dots, k$;

(C2) $a_A = 0 \pmod 2$, if $|A| \geq 2$;

(C3) $a_A = 0 \pmod 4$, if $|A| \geq 2$, $A \nsubseteq N$;

(C4) $a_{N_m} + 2a_N = 0$, for $m = 1, \dots, k$;

(C5) $a_{N_m} = 0$ for $m = 1, \dots, k$;

(C6) $a_{N_1} + \cdots + a_{N_k} + 2a_N = 0$;

(C7) $a_{N_1} + \cdots + a_{N_k} = 0$;

(C8) $a_{N_1} = \cdots = a_{N_k}$;

(C9) $2a_N = 0$;

where $N = \{1, 3, 5, \dots, 2k-1\}$, $N_m = \{\{1, 3, 5, \dots, 2k-1\} - \{2m-1\}\} \cup \{2m\}$ for $m = 1, \dots, k$.

Definition

Let be $n = 4, 5, 6, 7, 8, 9$, then H_n is $2^{2^{k}}$-ary relation on $\mathbb{Z}_8$, that consists of all elements satisfying (C1), (C2), (C3) a (Cn).

	H_4	H_5	H_8	H_9
$2t_n$	$n < k$	$n < k$	$n < k+1$	$n < \infty$
$2u_n$	$n < k$	$n < k$	$n < k$	$n < \infty$
$2v_n$	$n < k$	$n < k$	$n < k$	$n < \infty$
$2s_n$	$n < k$	$n < \infty$	$n < \infty$	$n < k$
$2p_n$	$n < k$	$n < k$	$n < k$	$n < k$
$2q_n$	$n < k-1$	$n < k-1$	$n < k-1$(*)	$n < k$
$2r_n$	$n < k+1$	$n < k$	$n < k+1$	$n < k$
$4r_n$	$n < \infty$	$n < \infty$	$n < \infty$	$n < \infty$

	H_6 k even	H_6 k odd	H_7 k even	H_7 k odd
$2t_n$	$n < k+1$	$n < k$	$n < k+1$	$n < k$
$2u_n$	$n < k$	$n < k$	$n < k$	$n < k$
$2v_n$	$n < k+1$	$n < k+1$	$n < k+1$	$n < k+1$
$2s_n$	$n < k$	$n < k$	$n < \infty$	$n < \infty$
$2p_n$	$n < k+1$	$n < k$	$n < k$	$n < k+1$
$2q_n$	$n < k$	$n < k$	$n < k$	$n < k$
$2r_n$	$n < k$	$n < k+1$	$n < k+1$	$n < k$
$4r_n$	$n < \infty$	$n < \infty$	$n < \infty$	$n < \infty$

Definition

Let be $2 \leq l \leq k$. $R_{k,l}$ is 2^k-ary relation on $\mathbb{Z}_8$, where all elements satisfy the following conditions

(D1) $a_A = 0 \pmod 2$, *if* $|A| \geq 2$;

(D2) $a_A = 0 \pmod 4$, *if* $|A| \geq l$;

(D3) $a_{\{1,\ldots,k\}} = 0$.

$2t_n$	$n < k - l + 2$ a $n < k - 1$
$2u_n$	$n < k - l + 2$ a $n < k - 1$
$2v_n$	$n < k - l + 2$ a $n < k - 1$
$2s_n$	$n < l$ a $n < k - 1$
$2p_n$	$n < k - l + 2$ a $n < l$
$2q_n$	$n < k - l + 1$, $n < k - 2$ a $n < l$
$2r_n$	$n < k - l + 2$ a $n < l$
$4r_n$	$n < k$

Thank you for your attention :)