# Boolean sublattices in partition lattices

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**Decomposition systems**

Let $(U, \mathcal{C})$ be a closure system. A decomposition in $(U, \mathcal{C})$ is a partition of the set $U$ whose blocks are closed sets from $\mathcal{C}$. If the closure system $(U, \mathcal{C})$ is algebraic and satisfies the property

$$\emptyset \in \mathcal{C} \text{ and } \{u\} \in \mathcal{C}, \text{ for all } u \in U. \quad (I_0)$$

then its decompositions form a complete lattice denoted by $\mathcal{D}(U, \mathcal{C})$, having as least element $\triangle := \{\{x\} \mid x \in U\}$.

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**Theorem**[FR1] $\mathcal{D}(U, \mathcal{C})$ is a complete sublattice of $\mathrm{Part}(U)$ if and only if $(U, \mathcal{C})$ satisfies $(I_0)$ and the condition:

$$\text{If } A, B \in \mathcal{C} \text{ and } A \cap B \neq \emptyset, \text{ then } A \cup B \in \mathcal{C}. \quad (I_1)$$

## Examples

If $T = (U; E)$ is a finite tree, then the vertex sets of its subtrees form a closure system $(U, \mathcal{C})$ which satisfies conditions $(I_0)$ and $(I_1)$. Then $\mathcal{D}(U, \mathcal{C})$ is a greatest possible Boolean sublattice of $\mathrm{Part}(U)$, having the size $2^{n-1}$, where $n = \mid U \mid$.
Closure systems satisfying conditions $(I_0)$ and $(I_1)$ are called weak interval systems. In their case $\mathcal{D}(U, \mathcal{C})$ is a semimodular sublattice of $\mathrm{Part}(U)$.

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## The questions examined here:

Which are the maximal Boolean sublattices of Part$(U)$ (w.r. to $\subseteq$)?
Which are the Boolean decomposition sublattices of Part$(U)$?

## 2. Preliminary observations

Clearly, the least element of Part($U$) is the partition $\triangle$ of $U$ having only blocks of the form $\{u\}$, $u \in U$, and its greatest element is the partition $\nabla$ having as a single block the whole set $U$.

**Lemma 1.**

Let $U \neq \emptyset$ be a finite set. Any Boolean sublattice $\mathcal{B}$ of Part($U$) can be extended into a Boolean sublattice $\mathcal{B}^*$ of Part($U$) with $\triangle, \nabla \in \mathcal{B}^*$.

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Let $\mathcal{B}$ be a Boolean sublattice of Part($U$) with $\triangle, \nabla \in \mathcal{B}$. As $\mathcal{B}$ is finite, it is an atomistic lattice - the set of its atoms will be denoted by $A(\mathcal{B})$.

**Lemma 2.**

If $\mathcal{B}$ is strictly included in a Boolean sublattice $\mathcal{D}$ of Part($U$), then there exists an atom $\alpha \in A(\mathcal{B})$ such that $\alpha$ is a proper join of some atoms of $\mathcal{D}$ strictly less than $\alpha$. Moreover, there exist some elements $\delta, \delta' \in \mathcal{D}$ with $\delta \vee \delta' = \alpha$, $\delta \wedge \delta' = \triangle$.

We will see that Lemma 2, in fact shows, how the Boolean sublattices of Part($U$) can be extended in some elementary steps.

## 3. Sublattices generated by disjoint and independent systems

(1) Let $H$ be a nonempty subset of a lattice $L$ with 0. A finite meet of elements in $H$ is called a *$\wedge$-element*. The set of all $\wedge$-elements of $H$ is denoted by $H_\wedge$. The set $H_\vee$ of $\vee$-elements of $H$ is defined dually.

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2) A nonempty subset $D = \{d_i \mid i \in I\} \subseteq L \setminus \{0\}$ is called a *disjoint system*, if for any $i, j \in I$, $i \neq j$, we have $d_i \wedge d_j = 0$.

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(3) $D \subseteq L \setminus \{0\}$ is called *independent* if the relation

$$\left(\bigvee X\right) \wedge \left(\bigvee Y\right) = \bigvee (X \cap Y) \tag{2}$$

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holds for all finite subsets $X, Y$ of $D$. Clearly, any independent subset of $L$ is also a disjoint system. In view of [Gr], if $D$ is a finite independent set of the lattice $L$, then the mapping $\varphi \colon X \to \bigvee X$, $X \subseteq D$ is an isomorphism between the Boolean lattice $\wp(D)$ of the subsets of $D$, and $\langle D \rangle$, the sublattice of $L$ generated by $D$. Clearly, now $\langle D \rangle = D_\vee$.

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**Lemma 3.**

Let $L$ be a finite lattice with 0, and $D \subseteq L \setminus \{0\}$ a disjoint system in $L$. Then the following assertions are equivalent:
(a) $D_\vee \cup \{0\}$ is a Boolean sublattice of $L$.
(b) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ holds for every $a, b, c \in D_\vee \cup \{0\}$.
(c) $D$ is independent in $L$.

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## 4. The hypergraph defined by a disjoint system of Part$(U)$

Let $D$ be a disjoint system in Part$(U) \setminus \{\triangle\}$. We define a hypergraph $\mathcal{H}(D) = (U, \mathcal{E})$ as follows: the *vertexes* of $\mathcal{H}(D)$ are the elements of $U$, and its *edge set* $\mathcal{E}$ (called also its *set of blocks*) consist from the nonsingleton blocks of all partitions $\pi \in D$. By definition $\mathcal{H}(D)$ do not contain loops or multiple edges, moreover, two different edges $A_1, A_2 \in \mathcal{E}$ can have at most one common vertex $v \in A_1 \cap A_2$.

These properties mean that $\mathcal{H}(D)$ is a *simple linear hypergraph.* Two different partitions $\pi_1, \pi_2 \in D$, $\pi_1 \neq \pi_2$ can have as common blocks only singletons.

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**Definition 1.**

The *incidence graph* of the hypergraph $\mathcal{H}(D)$ is the bipartite graph with vertex set $U \cup \mathcal{E}$ in which $x \in U$ is adjacent to $B \in \mathcal{E}$ iff $x \in B$.
(i) A *point-to-point path* in the hypergraph $\mathcal{H}(D)$ is a path in the incidence graph starting and ending in points.
(ii) A *cycle* $C$ in $\mathcal{H}(D)$ is a simple circuit in its incidence graph, i.e. it is a sequence $v_1, B_1, v_2, B_2, v_3, ..., v_n, B_n, v_1$ , where $v_1 \in B_1 \cap B_n$, $v_i \in B_{i-1} \cap B_i$, $2 \leq i \leq n$,($n \geq 3$), and all its points $v_1, ..., v_n \in U$ and all its edges $B_1, ..., B_n \in \mathcal{E}$ are different.

**Proposition 1.**

Let $\mathcal{B}$ be a Boolean sublattice of Part$(U)$ with $\triangle \in \mathcal{B}$, and $A(\mathcal{B})$ its set of atoms. Then in any cycle $\mathcal{C} := v_1, B_1, v_2, B_2, v_3, ..., v_n, B_n, v_1$ in $\mathcal{H}(A(\mathcal{B}))$, whose edges $B_1, ..., B_n$ are (nonsingleton) blocks of the partitions $\alpha_1, ..., \alpha_k \in A(\mathcal{B})$, each $\alpha_1, ..., \alpha_k$ participates with at least two blocks.

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## 5. The main results

Let $D$ be a disjoint system of Part$(U) \setminus \{\triangle\}$, and let us consider now the hypergraph $\mathcal{H}(D) = (U, \mathcal{E})$, defined previously.

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## Condition (C)

Let $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \in D$ and $v_1, A_1, v_2, A_2, \ldots, v_n A_n v_{n+1}$ respectively $w_1, B_1, w_2, B_2, \ldots, w_m B_m w_{m+1}$, be two paths in $\mathcal{H}(D)$ with common endpoints $v_1 = w_1 \in A_1 \cap B_1$ and $v_{n+1} = w_{m+1} \in A_n \cap B_m$, formed by the blocks $A_1, \ldots, A_n$ of some partitions from $\{\alpha_1, \ldots, \alpha_p\}$ and by the blocks $B_1, \ldots, B_m$ of some partitions from $\{\beta_1, \ldots, \beta_q\}$, and assume $\{\alpha_1, \ldots, \alpha_p\} \cap \{\beta_1, \ldots, \beta_q\} = \{\gamma_1, \ldots, \gamma_r\}$. We say that the hypergraph $\mathcal{H}(D)$ *satisfies condition (C)*, if there exists a path $u_1, C_1, u_2, C_2 \ldots, u_k, C_k, u_{k+1}$ with $u_1 = v_1 \in C_1$ and $u_{k+1} = v_{n+1} \in C_k$ in $\mathcal{H}(D)$ such that $C_1, \ldots, C_k$ are blocks of some partitions from $\{\gamma_1, \ldots, \gamma_r\}$.

**Theorem 1.**

Let $U \neq \emptyset$ be a finite set and $D \subseteq \text{Part}(U) \setminus \{\triangle\}$ is a disjoint system. Then $\mathcal{L} = D_{\vee} \cup \{\triangle\}$ is a Boolean sublattice of $\text{Part}(U)$ if and only the hypergraph $\mathcal{H}(D)$ induced by $D$ satisfies condition (C).

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For any subset $A \subseteq U$ with at least two elements, we denote by $\pi_A$ the principal partition: $\pi_A = \{A\} \cup \{\{x\} \mid x \in U \setminus A\}$.

**Theorem 2.**

Let $\triangle \in \mathcal{B}$ be a Boolean sublattice of a finite $\mathrm{Part}(U)$, and $\alpha$ an atom in $\mathcal{B}$. Then replacing $\alpha$ by the partitions $\{\pi_A \mid A$ is a nonsingleton block of $\alpha\}$, the sublattice of $\mathrm{Part}(U)$ generated by $(A(\mathcal{B}) \setminus \{\alpha\}) \cup \{\pi_A \mid A \in \alpha\}$ is a Boolean sublattice of $\mathrm{Part}(U)$ if and only if the hypergraph $\mathcal{H}(A(\mathcal{B}))$ does not contain a cycle that includes any block of $\alpha$.

**Corollary 1.**

Let $U \neq \emptyset$ be a finite set and $\mathcal{B}$ be a Boolean sublattice of $\mathrm{Part}(U)$ with $\triangle \in \mathcal{B}$. Then $\mathcal{B}$ equals to a decomposition lattice $\mathcal{D}(U, \mathcal{C})$ of a closure system $(U, \mathcal{C})$ satisfying condition $(I_0)$ if and only if the hypergraph $\mathcal{H}(A(\mathcal{B}))$ contains no cycles at all. In particular, if $\nabla \in \mathcal{B}$ then $\mathcal{H}(A(\mathcal{B}))$ is a *hypertree*.

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**Corollary 2.**

Let $\mathcal{B}$ be a Boolean sublattice of a finite Part$(U)$ with $\triangle, \nabla \in \mathcal{B}$. Then $\mathcal{B}$ equals to a maximal decomposition lattice $\mathcal{D}(U, \mathcal{C})$, if and only if the closure system $\mathcal{C}$ consists from the subtrees of a tree with vertex set $U$.

- M.K. Bennett, Lattices and geometry, in: Lattice theory and its applications, K.A. Baker and R. Wille (eds.) Heldermann Verlag, (1995) 27-50.
- J.E. Bonin and K.P. Bogart, A geometric characterisation of Dowling lattices, J. Comb. Thy A, **56** (1991), 195-202.
- A. Bretto, Hypergraph theory, An introduction. Mathematical Engineering. Cham: Springer, 1, (2013), 209-216.
- S. Foldes and S. Radeleczki, On interval decomposition lattices, Discussiones Mathematicae, General Algebra and Applications, **24** (2004), 95-114.
- S. Foldes and S. Radeleczki, Interval lattices are balanced, Demonstratio Mathematica, **49**(3) (2016), 271-281.
- G. Grätzer General Lattice Theory: Foundation, Birkhäuser/Springer, Basel, 2011

E. K. Horváth and S. Radeleczki, Notes on CD-independent subsets, Acta Sci. Math. (Szeged) **78** (2012), 3-24.

J. Kahn and J. Kung, A classification of modularly complemented geometric lattices, European J. of Comb. **7** (1986), 243-248.

R. H. Möhring and F. J. Radermacher, Substitution decomposition of discrete structures and connections to combinatorial optimization, Ann. Discrete Math. **19** (1984), 257-355.

M. Stern, Semimodular Lattices, Theory and Applications, Cambridge University Press, 1999.

S. Tamura, A note on distributive sublattices of a lattice, Proc. Japan Acad., **47** (1971), 603-605.

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