Categorical Equivalence between Finitary Orthomodular Dynamic Algebras and Orthomodular Lattices

Juanda Kelana Putra and Richard Smolka

Department of Mathematics and Statictics
Masaryk University, Brno, Czech Republic.

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Outline

Introduction

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- Kishida, Rad, Sack and Zhong (2017) then modify the definition of Quantum dynamic algebra to ensure categorical equivalences with **Complete orthomodular lattices**.

- In this talk, we extend the results of Kishida et al. by defining a **Finitary orthomodular dynamic algebra**, wherein the associated quantum actions are finite, and subsequently prove that it is categorically equivalent to **Orthomodular lattices**.

Ordered sets preliminaries

Definition 1

An **ortholattice** is a bounded lattice $\mathcal{M} = (M, \leq, 0, 1)$ equipped with an orthocomplementation map $\perp: M \to M$ that satisfies the following axioms for all $m, n \in M$:

1. **Complementation:** $m \wedge m^{\perp} = 0$ and $m \vee m^{\perp} = 1$.
2. **Involution:** $(m^{\perp})^{\perp} = m$.
3. **Order-Reversing:** If $m \leq n$, then $n^{\perp} \leq m^{\perp}$.

An **orthomodular lattice** is an ortholattice that satisfies the **orthomodular law**: for all $m, n \in M$ such that $m \leq n$, it holds that

$$n = m \vee (m^{\perp} \wedge n).$$

Ordered sets preliminaries

Definition 2

Given two orthomodular lattices $\mathcal{M}_1 = \left(M_1, \leq_1, -^{\perp_1}\right)$ and $\mathcal{M}_2 = \left(M_2, \leq_2, -^{\perp_2}\right)$, an **ortholattice isomorphism** $g : \mathcal{M}_1 \to \mathcal{M}_2$ is a function $g : M_1 \to M_2$ that satisfies the following conditions for every $m, n \in M_1$:

1. **Bijectivity:** g is a bijection.
2. **Order-Preservation:** $m \leq_1 n \Leftrightarrow g(m) \leq_2 g(n)$.
3. **Orthocomplementation-Preservation:** $g\left(m^{\perp_1}\right) = (g(m))^{\perp_2}$.

$\mathbb{OML}$ is the category of orthomodular lattices and ortholattice isomorphisms.

Definition 3

An **m-semilattice** is a tuple $\mathcal{K} = (K, \sqcup, \odot)$ satisfying the following:

1. $(K, \sqcup)$ is a **bounded join-semilattice**.
2. $(K, \odot)$ is a **semigroup**.
3. $\odot$ **distributes** over finite joins in K.

An m-semilattice is **unital** if its semigroup is a **monoid** (i.e., has an identity element).

An m-semilattice $\mathcal{K}$ is **involutive** if there exists a unary operation $-^*$ on K such that $(K, \odot, -^*)$ forms an **involutive semigroup** and the involution distributes over finite joins in K.

Remark 1

- Within an ortholattice $\mathcal{M} = (M, \leq, ^{\perp})$, the **Sasaki projection** onto an element $m \in M$ is the map $\pi_m : M \to M$ defined by $\pi_m(n) = m \wedge (m^{\perp} \vee n)$ for all $n \in M$.
- For orthomodular lattices $\mathcal{M}$ and $\mathcal{N}$, a function $f : M \to N$ is defined as a **linear map** if there exists a map $f^* : N \to M$, called its **adjoint**, such that the following condition holds for all $m \in M$ and $n \in N$:

$$f(m) \perp n \quad \Longleftrightarrow \quad m \perp f^*(n).$$

The set of all linear maps from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $\mathbf{Lin}(\mathcal{M}, \mathcal{N})$. When $\mathcal{M} = \mathcal{N}$, the set $\mathbf{Lin}(\mathcal{M}, \mathcal{M}) = \mathbf{Lin}(\mathcal{M})$ is a unital involutive m-semilattice.

Finitary generalized dynamic algebra

Definition 4

A **finitary generalized dynamic algebra** is a tuple $\mathfrak{K} = (K, \bigsqcup, \odot, \sim)$ where:

1. K is a non-empty set.
2. $\bigsqcup : \mathscr{P}_{\text{fin}}(K) \to K$ is a finitary join operation, where $\mathscr{P}_{\text{fin}}(K)$ denotes the set of all finite subsets of K.
3. $\odot : K \times K \to K$ is a binary operation.
4. $\sim : K \to K$ is a unary operation.

From this, we derive the following terms and constructions:

Terms and constructions on $\mathfrak{K} = (K, \bigsqcup, \odot, \sim)$:

- **Complemented Elements:** The set of complemented elements, denoted by $\widetilde{K}$, is defined as:

$$\widetilde{K} \stackrel{\text{def}}{=} \{\sim k \mid k \in K\}$$

- **Finite Join:** The finite join of a set $W \subseteq \widetilde{K}$ is defined as:

$$\bigvee W \stackrel{\text{def}}{=} \sim (\sim \bigsqcup W), \quad \text{for any finite } W \subseteq \widetilde{K}$$

- **Finite Meet:** The finite meet of a set $W \subseteq \widetilde{K}$ is defined as:

$$\bigwedge W \stackrel{\text{def}}{=} \sim \bigsqcup \{\sim w \mid w \in W\}, \quad \text{for any finite } W \subseteq \widetilde{K}$$

- **Order Relation:** The order relation $\preceq$ on $\widetilde{K}$ is defined as:

$$\preceq \stackrel{\text{def}}{=} \left\{(k, l) \in \widetilde{K} \times \widetilde{K} \mid \bigvee \{k, l\} = l\right\}$$

- **Generated Elements:** The set of elements generated by complemented elements via the $\odot$ operation, denoted by $\langle \widetilde{K} \rangle$, is defined as:

$$\left\langle \widetilde{K} \right\rangle \stackrel{\text{def}}{=} \left\{ k \in K : k = w_1 \odot \cdots \odot w_n, \text{ for some } n \in \mathbb{N}^+ \text{ and } w_1, \ldots, w_n \in \widetilde{K} \right\}.$$

- **Dynamic Closure:** For a fixed element $k \in K$, a unary operation is defined as:

$$\ulcorner k \urcorner (l) \stackrel{\text{def}}{=} \sim (\sim (k \odot l)), \quad \text{for all } l \in K$$

- **Dynamic Equivalence Axiom:** The Dynamic Equivalence Axiom introduces an equivalence relation $\equiv$ on K:

$$\equiv \stackrel{\text{def}}{=} \left\{ (k, l) \in K \times K \mid \ulcorner k \urcorner (w) = \ulcorner l \urcorner (w), \text{ for every } w \in \widetilde{K} \right\}$$

Finitary orthomodular dynamic algebra

Definition 5

A **finitary orthomodular dynamic algebra** is a finitary generalized dynamic algebra $\mathfrak{K} = (K, \bigsqcup, \odot, \sim)$ with a unary operation $-^* : K \to K$ that satisfies the following conditions:

1. $\mathfrak{K}$ forms a **unital involutive m-semilattice**, and $(\widetilde{K}, \preceq, \sim)$ is an **orthomodular lattice** where $x^* = x$ for all $x \in \widetilde{K}$.
2. **Finitary Uniqueness Axiom:** For any subsets $S, T \subseteq \left\langle \widetilde{K} \right\rangle$, $\bigsqcup S = \bigsqcup T$ if and only if $S = T$.
3. **Dynamic Equality Axiom:** For any $s, t \in \left\langle \widetilde{K} \right\rangle$, $s = t$ if and only if $s \equiv t$.

Finitary orthomodular dynamic algebra

4. **Generative Axiom:** K is the smallest subset of itself that contains $\widetilde{K}$ and is closed under the $\odot$, $-^*$, and $\bigsqcup$ operations.
5. **Dynamic Orthomodularity Axiom:** For any $v, w \in \widetilde{K}$, $\ulcorner v \urcorner(w) = \pi_v(w)$, where π_v is an Sasaki projection.
6. **Composition Axiom:** For each $k, l \in K$, the composition property holds: $\ulcorner k \urcorner(l) = \ulcorner k \urcorner(\sim(\sim l))$.

Definition 6

A $\mathbb{FODA}$-**morphism** $f : \mathfrak{K}_1 \to \mathfrak{K}_2$ is a function between two finitary orthomodular dynamic algebras that preserves their structure. Specifically, for all elements and finite sets in the domain, it satisfies:

1. **Ortholattice Isomorphism Axiom:** $f|_{\widetilde{K_1}} : \widetilde{K}_1 \to \widetilde{K}_2$ is an ortholattice isomorphism.
2. **Finitary Join Preservation:** f preserves the finitary join operation, i.e., $f\left(\bigsqcup_1 A\right) = \bigsqcup_2 \{f(a) \mid a \in A\}$.
3. **Dynamic Product Preservation:** f preserves the dynamic product operation, i.e., $f(k \odot_1 l) = f(k) \odot_2 f(l)$.

4. **Dynamic Complement Preservation:** f preserves the dynamic complement operation, i.e., $f\left(\sim_1 k\right) = \sim_2 \left(f(k)\right)$.
5. **Involution Preservation:** f preserves the involution operation, i.e., $f\left(k^{*_1}\right) = \left(f(k)\right)^{*2}$.
6. **Unit Preservation:** f maps the unit element of the first algebra to the unit element of the second, i.e., $f(e_1) = e_2$.

$\mathbb{FODA}$ is the category of finitary orthomodular dynamic algebras and $\mathbb{FODA}$-morphisms.

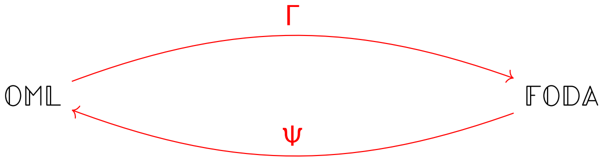
We will now prove that the category of orthomodular lattices ($\mathbb{OML}$) is categorical equivalence to the category of finitary orthomodular dynamic algebras ($\mathbb{FODA}$).

Categorical equivalence

Definition 7

An equivalence between categories $\mathbb{C}$ and $\mathbb{D}$ is a pair of covariant functors $\Gamma : \mathbb{C} \to \mathbb{D}$ and $\Psi : \mathbb{D} \to \mathbb{C}$ such that

1. there is a natural isomorphism $\mu : 1_{\mathbb{C}} \to \Psi \circ \Gamma$
2. there is a natural isomorphism $\lambda : 1_{\mathbb{D}} \to \Gamma \circ \Psi$.



The Functor $\Gamma : \mathbb{OML} \to \mathbb{FODA}$

Mapping of Objects

For an arbitrary orthomodular lattice $\mathcal{M} = (M, \leq, -^{\perp})$, The object mapping of Γ is defined such that $\Gamma(\mathcal{M}) = \left(\mathscr{P}_{\text{fin}}(S_{\mathcal{M}}), \bigcup, \odot, \sim, -^{*}\right)$, where:

1. $S_{\mathcal{M}}$: The smallest subset of containing all **Sasaki projections** on M and closed under function composition and involution.
2. $\mathscr{P}_{\text{fin}}(S_{\mathcal{M}})$: The set of all finite subsets of $S_{\mathcal{M}}$.
3. **Dynamic Product** $(\odot)$: A binary operation on $\mathscr{P}_{\text{fin}}(S_{\mathcal{M}})$ defined by

$$A \odot B \;=\; \{\, a \circ b \mid a \in A,\; b \in B \} \in \mathscr{P}_{\text{fin}}(S_{\mathcal{M}})$$

for every $A, B \in \mathscr{P}_{\text{fin}}(S_{\mathcal{M}})$.

The Functor $\Gamma : \mathbb{OML} \to \mathbb{FODA}$

4. **Dynamic Complement** ($\sim$): A unary operation on $\mathscr{P}_{\text{fin}}(S_{\mathcal{M}})$ defined by

$$\sim A = \left\{ \pi_{\left(\bigvee \{ a(1) \mid a \in A \} \right)^{\perp}} \right\}$$

for every $A \in \mathscr{P}_{\text{fin}}(S_{\mathcal{M}})$.

5. **Involution** $(-^{*})$: A unary operation on $\mathscr{P}_{\text{fin}}(S_{\mathcal{M}})$ defined by

$$A^{*} = \{ a^{*} \mid a \in A \}$$

for every $A \in \mathscr{P}_{\text{fin}}(S_{\mathcal{M}})$.

The Functor $\Gamma : \mathbb{OML} \to \mathbb{FODA}$

Mapping of Arrows

Let $\mathcal{M}_1 = (M_1, \leq_1, -^{\perp_1})$ and $\mathcal{M}_2 = (M_2, \leq_2, -^{\perp_2})$ be orthomodular lattices. Given an ortholattice isomorphism $g : \mathcal{M}_1 \to \mathcal{M}_2$, the arrow mapping of Γ is defined as

$$\Gamma(g) : \Gamma(\mathcal{M}_1) \to \Gamma(\mathcal{M}_2)$$

$$A \mapsto \{g \circ a \circ g^{-1} \mid a \in A\}$$

Theorem 8

The mapping Γ constitutes a functor from the category of orthomodular lattices ($\mathbb{OML}$) to the category of finitary orthomodular dynamic algebras ($\mathbb{FODA}$).

The Functor $\Psi : \mathbb{FODA} \to \mathbb{OML}$

Mapping of Objects

Let $\mathfrak{K} = (K, \bigsqcup, \odot, \sim, -^*)$ be a finitary orthomodular dynamic algebra. The object mapping of Ψ is defined such that $\Psi(\mathfrak{K}) = (\widetilde{K}, \preceq, -^{\perp})$, where $\widetilde{K} = \{\sim k \mid k \in K\}$.

Mapping of Arrows

Let $\mathfrak{K}_1 = (K_1, \bigsqcup_1, \odot_1, \sim_1, -^{*_1})$ and $\mathfrak{K}_2 = (K_2, \bigsqcup_2, \odot_2, \sim_2, -^{*_2})$ be finitary orthomodular dynamic algebras. Given a $\mathbb{FODA}$-morphism $f : \mathfrak{K}_1 \to \mathfrak{K}_2$, the arrow mapping of Ψ is defined as

$$(\Psi)(f) : \Psi(\mathfrak{K}_1) \to (\Psi)(\mathfrak{K}_2)$$

$$k \mapsto f(k)$$

The Functor $\Psi : \mathbb{FODA} \to \mathbb{OML}$

Theorem 9

The mapping Ψ constitutes a functor from the category of finitary orthomodular dynamic algebras ($\mathbb{FODA}$) to the category of orthomodular lattices ($\mathbb{OML}$).

To establish a **categorical equivalence** between $\mathbb{OML}$ and $\mathbb{FODA}$, we now define the natural isomorphisms $\mu : 1_{\mathbb{OML}} \to \Psi \circ \Gamma$ and $\lambda : 1_{\mathbb{FODA}} \to \Gamma \circ \Psi$.

The Natural Isomorphism $\lambda : 1_{\mathbb{FODA}} \to \Gamma \circ \Psi$

The Natural Isomorphism $\lambda : 1_{\mathbb{FODA}} \to \Gamma \circ \Psi$

We define the natural transformation $\lambda : 1_{\mathbb{FODA}} \to \Gamma \circ \Psi$ such that $\lambda_{\mathfrak{K}} : \mathfrak{K} \to (\Gamma \circ \Psi)(\mathfrak{K})$ for every object $\mathfrak{K} = (K, \bigsqcup, \odot, \sim, -^*)$ in $\mathbb{FODA}$ as:

$$\lambda_{\mathfrak{K}}(k) = \left\{\pi_{p_{(i,1)}} \circ \pi_{p_{(i,2)}} \circ \ldots \circ \pi_{p_{(i,n_i)}} \mid 1 \leq i \leq m\right\}$$

for every $k = \bigsqcup_{i=1}^{m} \{p_{(i,1)} \odot p_{(i,2)} \odot \ldots \odot p_{(i,n_i)} | \; 1 \leq i \leq m\} \in K$,

where $p_{(i,j)} \in \widetilde{K}$ for each $(i,j) \in \bigcup_{i=1}^{m} (\{i\} \times \{1, \ldots, n_i\})$ and $(\Gamma \circ \Psi)(\mathfrak{K}) = \mathscr{P}_{\text{fin}}(S_{\widetilde{K}})$.

This natural transformation λ constitutes a natural isomorphism.

The Natural Isomorphism $\mu : 1_{\mathbb{OML}} \to \Psi \circ \Gamma$

The Natural Isomorphism $\mu : 1_{\mathbb{OML}} \to \Psi \circ \Gamma$

We define the natural transformation $\mu : 1_{\mathbb{OML}} \to \Psi \circ \Gamma$ as

$$\mu_{\mathcal{M}} : \mathcal{M} \to (\Psi \circ \Gamma)(\mathcal{M})$$

$$m \mapsto \{\pi_m\}$$

for every object $\mathcal{M} = (M, \leq, -^{\perp})$ in $\mathbb{OML}$, where

$(\Psi \circ \Gamma)(\mathcal{M}) = \{\sim W \mid W \in \mathscr{P}_{\text{fin}}(S_{\mathcal{M}})\}$

This natural transformation μ constitutes a natural isomorphism.

Theorem 10

The quadruple $(\Gamma, \Psi, \lambda, \mu)$ establishes a categorical equivalence between $\mathbb{OML}$ and $\mathbb{FODA}$.

Conclution and future research

Conclution and future research

- The category $\mathbb{FODA}$ is categorically equivalent to the category $\mathbb{OML}$.
- Future work will extend the finitary orthomodular dynamic algebra to **Relation-based orthomodular dynamic algebra** and **Function-based orthomodular dynamic algebra**, as described in Rad et al. (2025). We will then investigate their categorical equivalence.

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