How generalized quasiorders appear in rectangular algebras

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Outline

Generalized quasiorders and generalized partial orders

Rectangular algebras

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Rectangular algebras

A nice property: When translations are enough!?

$f : A^n \to A$ (n-ary operation), $\varrho \subseteq A^m$ (m-ary relation)

$$\forall f\colon \boxed{f \triangleright \varrho \iff \mathsf{trl}(f) \triangleright \varrho} \qquad (\Xi)$$

▸ Def ▹

$$\mathsf{trl}(f) := \{f(c_1, \ldots, c_{i-1}, x, c_{i+1}, \ldots, c_n) \mid i \in \{1, \ldots, n\}, c_1, \ldots, c_n \in A\}$$

(*unary translations* of f)

(Ξ) holds for

- equivalence relations $\varrho \in \mathrm{Eq}(A)$ (reflexiv, symmetric, transitive)
- quasiorder relations $\varrho \in \mathrm{Quord}(A)$ (binary, reflexive, transitive)
- generalized quasiorders (m-ary, reflexive, transitive)
  JPR 2022 (published in Algebra Universalis 2024)

history (JPR = D. Jakubíková-Studenovská, R.P., S. Radeleczki): investigation of (the lattice of) congruence and quasiorder lattices $\mathrm{Con}(A, F)$, $\mathrm{Quord}(A, F)$, (since 2007)

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Generalized quasiorders

Definition

Let $\varrho \subseteq A^m$ (m-ary relation)

- *reflexive* $:\iff \forall a \in A : (a, \dots, a) \in \varrho$.
- *transitive*
  $:\iff \forall (a_{ij})_{i,j \in \{1,\dots,m\}} : \varrho \models (a_{ij}) \implies (a_{11}, \dots, a_{mm}) \in \varrho$

- *generalized quasiorder* $:\iff$ reflexive & transitive
- $\mathrm{gQuord}(A) :=$ all generalized quasiorders on A ($m \in \mathbb{N}_+$)

Remark: $\mathrm{gQuord}^{(2)}(A) = \mathrm{Quord}(A)$

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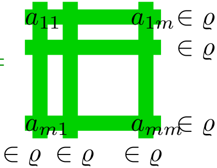

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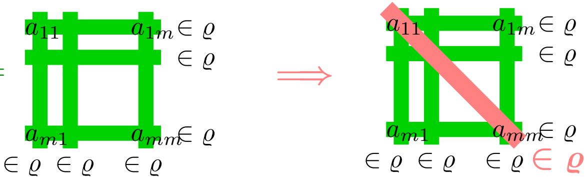

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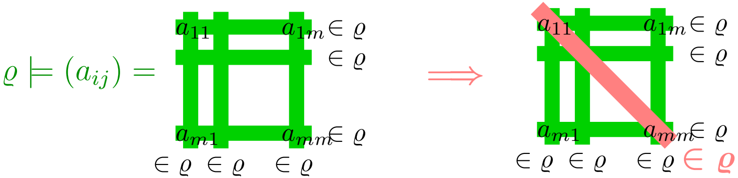

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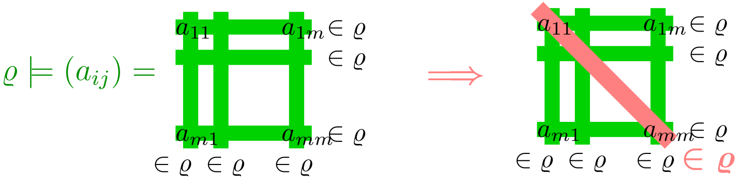

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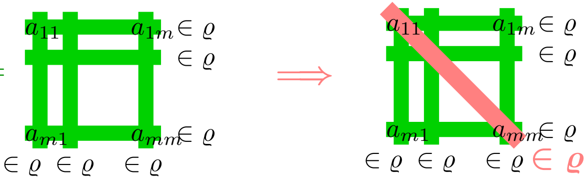

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Generalized partial orders

Partial orders $\varrho \subseteq A^2$ on A (*reflexive, antisymmetric, transitive*) clearly are quasiorders. Antisymmetry $((x,y),(y,x) \in \varrho \implies x = y)$ is equivalent to

(i) $\operatorname{tos}(\varrho) := \{(a_1,a_2) \in A^2 \mid \forall \pi \in \mathrm{Sym}(2) : (a_{\pi 1}, a_{\pi 2}) \in \varrho\} = \Delta_A$
(totally symmetric part of ϱ is trivial)

(ii) $\varrho^{[2]} := \{(a,b) \in A^2 \mid \{a,b\}^2 \subseteq \varrho\} = \Delta_A$
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Observation: here $\operatorname{tos}(\varrho) = \varrho^{[2]} = \varrho \cap \varrho^{-1}$ (symmetric part of ϱ)
Generalization:
A generalized quasiorder (reflexive, transitive) $\varrho \subseteq A^m$ is a *generalized partial order* if it satisfies one of the following equivalent conditions:

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Rectangular algebras

Rectangular bands

A *rectangular band* is a semigroup $(A, *)$ satisfying

$$x * x \approx x \qquad \text{(idempotence)}$$

$$x * y * z \approx x * z \qquad \text{(absorption)}$$

Proposition

Let $(A, *)$ be a rectangular band. Then the graph of $*$

$$\varrho := \{(a_1, a_2, b) \in A^3 \mid a_1 * a_2 = b\}$$

is a ternary generalized partial order.

Generalization: Rectangular algebras

Definition (cf.,e.g., [PösR1993])

An algebra $(A,(f_i)_{i\in I}) = (A,F)$ (of finite type) is called *rectangular algebra* if for all fundamental operations $f, g \in F$ (f n-ary, g m-ary) the following identities are satisfied:

(**ID**$_f$) $f(x,x,\dots,x) \approx x$ (idempotence)

(**AB**^i_f) $f(x_1,\dots,x_{i-1},f(y_1,\dots,y_{i-1},x_i,y_{i+1},\dots,y_n),x_{i+1},\dots,x_n) \approx f(x_1,...,x_n)$

(absorption in each place $i \in \{1,\dots,n\}$)

(**C**$_{f,g}$) $f(g(x_{11},\dots,x_{1m}),\dots,g(x_{n1},\dots,x_{nm})) \approx g(f(x_{11},\dots,x_{n1}),\dots,f(x_{1m},\dots,x_{nm}))$

(commuting operations)

Remark: if f is idempotent, then the absorption identities together are equivalent to the following single identity

(**AB**$_f$) $f(f(x_{11},\dots,x_{1n}),f(x_{21},\dots,x_{2n}),\dots,f(x_{n1},\dots,x_{nn})) \approx f(x_{11},\dots,x_{nn})$.

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Generalized partial orders in rectangular algebras

Proposition

(i) *Let* $f : A^n \to A$ *satisfy* $(\mathbf{ID}_f)$ *and* $(\mathbf{C}_{f,f})$.
Then f *satisfies* $(\mathbf{AB}_f)$ *if and only if the graph* $f^\bullet$ *of* f,

$$f^\bullet := \{(a_1, \dots, a_n, b) \in A^{n+1} \mid f(a_1, \dots, a_n) = b\},$$

is an $(n+1)$*-ary generalized quasiorder.*

(ii) *The graph* $t^\bullet$ *of each term operation* t *of a rectangular algebra* (A, F) *is a generalized partial order.*

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Proof

(i): Let $f^\bullet \models M$ for a matrix $M = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & b_n \\ c_1 & \dots & c_n & d \end{pmatrix}$,

Thus $f(a_{i1}, \dots, a_{in}) = b_i$ and $f(a_{1i}, \dots, a_{ni}) = c_i$ for $i \in \{1, \dots, n\}$ (first n rows and columns).

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Therefore, in M, the a_{ij} can be chosen arbitrarily.

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Proof (continued)

(ii): The variety of rectangular algebras is a so-called *solid variety*, i.e., each identity for the fundamental operations is also an identity for arbitrary term operations (of the corresponding arities).

Thus, in particular, each term operation t of a rectangular algebra satisfies the identities ($\mathbf{ID}_t$), ($\mathbf{C}_{t,t}$) and ($\mathbf{AB}_t$).

From (i) we can conclude that $t^\bullet$ is a generalized quasiorder.
it remains to show that it is a generalized partial order

Note $\{a,b\}^{n+1} \in f^\bullet$ implies $(a,\dots,a,a),(a,\dots,a,b) \in f^\bullet$,
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back


Thank you for your ATTENTION!

Function f preserves relation ϱ

function f (n-ary) preserves relation ϱ (m-ary): $f \triangleright \varrho$

$$\begin{array}{lcccccl} f(& a_{11} & a_{12} & \dots & a_{1n} &) = & \bullet \\ f(& a_{21} & a_{22} & \dots & a_{2n} &) = & \bullet \\ & & & & & & \\ f(& a_{m1} & a_{m2} & \dots & a_{mn} &) = & \bullet \\ & \in \varrho & \in \varrho & \dots & \in \varrho & \Rightarrow & \in \varrho \end{array}$$

$F \subseteq \mathrm{Op}(A)$ (set of all finitary operations $f : A^n \to A$)
$Q \subseteq \mathrm{Rel}(A)$ (set of all finitary relations $\varrho \subseteq A^m$)

$\mathrm{Inv}\, F := \{\varrho \in R_A \mid \forall f \in F : f \triangleright \varrho\}$ *invariant relations*

$\mathrm{Pol}\, Q := \{f \in \mathrm{Op}(A) \mid \forall \varrho \in Q : f \triangleright \varrho\}$ *polymorphisms*

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