Foulis Quantales

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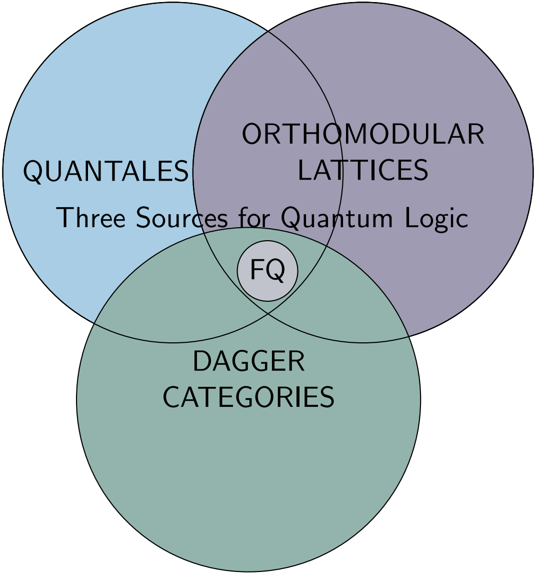
Outline

1. Introduction
2. Algebraic and categorical preliminaries
3. Category **SupOMLatLin** of complete orthomodular lattices
4. Foulis quantales and complete orthomodular lattices
5. Conclusion


QUANTALES
ORTHOMODULAR LATTICES
Three Sources for Quantum Logic
FQ
DAGGER CATEGORIES

Lecture Content - Preliminaries

- We will introduce quantales, algebraic structures that generalize frames and play a crucial role in various areas of mathematics, including domain theory and logic. We will discuss their key properties.
- We will then turn our attention to orthomodular lattices, algebraic structures that have been extensively studied in the context of quantum mechanics.
- Finally, we will introduce the concept of dagger categories, a categorical framework that provides a powerful tool for studying quantum systems. Dagger categories are equipped with a special involution (the dagger) that allows for the representation of physical processes, including measurements and state transformations.

What are Quantales?

- A quantale is a complete lattice Q equipped with an associative binary operation $\cdot$ (called the multiplication) that distributes over arbitrary suprema:

$$a \cdot \bigsqcup_{i\in I} b_i = \bigsqcup_{i\in I} (a \cdot b_i) \qquad \text{and} \qquad \left(\bigsqcup_{i\in I} a_i\right) \cdot b = \bigsqcup_{i\in I} (a_i \cdot b)$$

- Think of it as a generalized notion of a "ring" where addition is replaced by arbitrary suprema and multiplication is not necessarily commutative.
- Quantales provide a powerful framework for studying various structures, including:
  - Frames (and hence topology)
  - Relations
  - Languages
  - ... and many more!

A Brief History of Quantales

- The name "quantale" itself was coined by **C.J. Mulvey** (1986) to emphasize the connection to "quanta" and the non-commutative nature of the multiplication.
- Quantales generalize locales and various multiplicative lattices of ideals from ring theory and functional analysis, such as C^*-algebras and von Neumann algebras.
- Significant contributions were made by **K.I. Rosenthal** and others (J.W. Pelletier, J. Rosický, J.P., D. Kruml, S. Abramsky, S. Vickers, P. Resende), who connected quantales to various areas like locales (and thus, pointless topology), C^*-algebras (study of spectra in C^*-algebras), and theoretical computer science (Linear Logic).

Examples of Quantales

- **The powerset of a monoid:** Let $(M, \cdot)$ be a monoid. The powerset $\mathcal{P}(M)$ forms a quantale where multiplication is given by
$$A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$$
and the join is given by set union.
- **The set of relations:** The set of all relations on a sets X, denoted by $\mathrm{Rel}(X)$, forms a quantale where multiplication is relation composition and the join is given by the union of relations.
- **Frames:** A frame is a complete lattice L where the following distributive law holds:
$$a \wedge \bigsqcup_{i \in I} b_i = \bigsqcup_{i \in I} (a \wedge b_i)$$
Frames are quantales where the multiplication is the meet operation $\wedge$.

Quantale Modules

- A (left) quantale module over a quantale Q is a complete lattice M equipped with an action of Q on M, denoted by $\bullet$, such that

$$a \bullet \bigvee_{i \in I} m_i = \bigvee_{i \in I} (a \bullet m_i) \qquad \text{and} \qquad \left(\bigsqcup_{i \in I} a_i \right) \bullet m = \bigvee_{i \in I} (a_i \bullet m)$$

$$(a \cdot b) \bullet m = a \bullet (b \bullet m)$$

for all $a, b \in Q$ and $m, m_i \in M$.

- Modules generalize the notion of vector spaces over fields to the setting of quantales.
- The definition of right Q-modules follows analogously. It is readily apparent that every complete lattice A is a right and left 2-module. Here, 2 is a 2-element chain, its multiplication is its meet and involution is the identity map on it.

Examples of Quantale Modules

- If Q is a quantale, then Q itself is a module over itself, where the action is the multiplication in Q.
- Let Q be a quantale and X be a set. The set of functions from X to Q, denoted by Q^X, is a Q-module where the action is given by pointwise multiplication:

$$(a \bullet f)(x) = a \cdot f(x)$$

for $a \in Q$, $f \in Q^X$, and $x \in X$.

Origins in Quantum Logic

- Orthomodular lattices emerged from Birkhoff and von Neumann's work (1936) on quantum mechanics
- They sought to understand the algebraic structure of quantum propositions
- Classical logic wasn't sufficient to capture quantum phenomena
- Led to development of quantum logic as an alternative to Boolean logic
- An ***orthomodular lattice*** is a bounded lattice $(L, \leq, 0, 1)$ with:
  - An ***orthocomplementation operation*** $'$ satisfying:
    - $x'' = x$
    - If $x \leq y$ then $y' \leq x'$
    - $x \wedge x' = 0$ and $x \vee x' = 1$
  - The ***orthomodular law***: if $x \leq y$ then $y = x \vee (y \wedge x')$
  - We write $x \perp y$ if and only if $x \leq y^{\perp}$.

Origins and Motivation, Definition

- Dagger categories emerged from mathematical physics in the 2000s
- Key early work by Abramsky and Coecke (2004)
- Developed to capture quantum mechanical structures categorically
- Provides abstract framework for quantum processes and protocols
- A ***dagger category*** is a category $\mathcal{C}$ equipped with a contravariant functor $(-)^\dagger : \mathcal{C}^{op} \to \mathcal{C}$ such that:
  - On objects: $A^\dagger = A$
  - On morphisms: $f^{\dagger\dagger} = f$, $\mathrm{id}_A^\dagger = \mathrm{id}_A$
  - Functoriality: $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$
- Think of $f^\dagger$ as an abstract adjoint/conjugate/reverse

Key Examples

Dagger categories are a categorical generalization of involutive semigroups in that involutive monoids are precisely the dagger categories with one object.

- Hilb: Category of Hilbert spaces
  - Morphisms are bounded linear operators
  - Dagger is the adjoint operator
- Rel: Category of sets and relations
  - Dagger is relation inverse
- $\mathrm{FdVect}_{\mathbb{C}}$: Finite-dimensional complex vector spaces
  - Dagger is conjugate transpose

Dagger categories

- Achievements: a purely categorical characterization of a complex Hilbert space by Heunen and Kornell.
- The present paper continues the study of dagger categories in relation to orthomodular lattices in the spirit of Jacobs [Jac].

Example 1 (Our guiding example)

The collection $\mathcal{C}(H)$ of closed subspaces of a Hilbert space H is the prototypical example of a complete orthomodular lattice such that $\wedge = \cap$ and $P^{\perp}$ is the orthogonal complement of a closed subspace P of H.

Linear maps

Definition 2

The category **SupOMLatLin** has complete orthomodular lattices as objects.
A morphism $f\colon X \to Y$ in **SupOMLatLin** is a function $f\colon X \to Y$ between the underlying sets such that there is a function $h\colon Y \to X$ and, for any $x \in X$ and $y \in Y$,

$$f(x) \perp y \text{ if and only if } x \perp h(y).$$

We say that h is an ***adjoint*** of a ***linear map*** f. It is clear that adjointness is a symmetric property: if a map f possesses an adjoint h, then f is also an adjoint of h, and that it is uniquely determined (we write f^* for g).
Moreover, a map $f\colon X \to X$ is called *self-adjoint* if f is an adjoint of itself.

Linear maps

The identity morphism on X is the self-adjoint identity map $\mathrm{id}\colon X \to X$. Composition of $X \xrightarrow{f} Y \xrightarrow{g} Z$ is given by usual composition of maps.

We denote $\mathbf{Lin}(X, Y)$ the set of all linear maps from X to Y. If $X = Y$ we put $\mathbf{Lin}(X) = \mathbf{Lin}(X, X)$. Evidently, $\mathbf{Lin}(X)$ is a semigroup with an involution.

Example 3 (Our guiding example - continuation)

Let $f\colon H_1 \to H_2$ be a bounded linear map between Hilbert spaces and let $f^\star$ be the usual adjoint of f given by $\langle f(x), y\rangle = \langle x, f^\star(y)\rangle$.

Then the induced map $C(H_1) \to C(H_2)$, $\langle S\rangle \mapsto \langle f(S)\rangle$ has the adjoint $C(H_2) \to C(H_1)$, $\langle T\rangle \mapsto \langle f^\star(T)\rangle$.

Properties of the category **SupOMLatLin**

Lemma 4

Let $f : X \to Y$ be a map between complete orthomodular lattices. The following three key properties of f are equivalent:

1. *f possesses a right order-adjoint;*
2. *f admits an adjoint in the sense of Definition 2;*
3. *f preserves arbitrary joins (i.e., is join-complete).*

This equivalence provides multiple perspectives for understanding linear maps in the context of complete orthomodular lattices.

Principal downsets in orthomodular lattices

Lemma 5

[Jac, Lemma 3.4] *Let X be an orthomodular lattice and $a \in X$. The (principal) downset $\downarrow a = \{u \in X \mid u \leq a\}$ is again an orthomodular lattice, with order, meets and joins as in X, but with its own orthocomplement $\perp_a$ given by $u^{\perp_a} = a \wedge u^{\perp}$, where $^{\perp}$ is the orthocomplement from X.*

Lemma 6 (Sasaki projection)

Let X be an orthomodular lattice and $a \in X$. There is a dagger monomorphism $\downarrow a \rightarrowtail X$ in **OMLatLin**, *for which we also write a, with $a(u) = u$ and $a^*(x) = \pi_a(x) = a \wedge (x \vee a^{\perp})$.*

Dagger category **SupOMLatLin**

The category of complete orthomodular lattices with linear maps is shown to constitute a dagger category by the following theorem.

Theorem 7

SupOMLatLin *is a dagger category. Here* $\dagger = *$.

Definition 8

1. A ***quantaloid*** is a locally small category whose hom-sets are complete lattices and whose composition preserves joins in both variables.
2. An ***involutive quantaloid*** is both a quantaloid and a dagger category $\mathbf{C}$ such that, for all $X, Y \in \mathbf{C}$ and all $S \subseteq Hom(X, Y)$,

$$(\bigvee S)^{\dagger} = \bigvee\{s^{\dagger} \mid s \in S\}.$$

Involutive quantales

Definition 9

By an ***involutive quantale*** will be meant a quantale Q together with a semigroup involution * satisfying

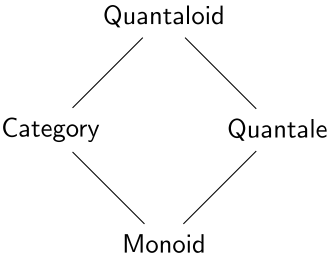
$$(\bigsqcup a_i)^* = \bigsqcup a_i^*$$

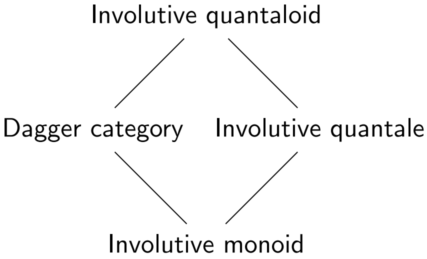
for all $a_i \in Q$. In the event that Q is also unital, then necessarily e is selfadjoint, i.e.,

$$e = e^*.$$

We denote by $\sqsubseteq$ the order relation on Q.
We also define $s \leq t$ if and only if $s = t \cdot s$, and $s \perp t$ if and only if $0 = s^* \cdot t$ for all $s, t \in Q$.

Involutive quantaloid **SupOMLatLin**





Theorem 10

SupOMLatLin *is an involutive quantaloid. Here* $\dagger = *$.

Foulis semigroups

Definition 11

A Foulis semigroup consists of a monoid $(S, \cdot, 1)$ together with two endomaps $(-)^*\colon S \to S$ and ${}^\perp\colon S \to S$ satisfying:

1. $1^* = 1$ and $(s \cdot t) = t^* \cdot s^*$ and $s^{**} = s$, making S an involutive monoid;
2. $s^\perp$ is a self-adjoint idempotent, i.e., $s^\perp \cdot s^\perp = s^\perp = (s^\perp)^*$;
3. $0 \overset{\text{def}}{=} 1^\perp$ is a zero element: $0 \cdot s = 0 = s \cdot 0$;
4. $s^* \cdot x = 0$ iff $\exists_y . x = s^\perp \cdot y$.

Theorem 12 ([Kal, Chapter 5, §§18])

Let X be a complete orthomodular lattice and let us define the endomap ${}^\perp\colon \mathbf{Lin}(X) \to \mathbf{Lin}(X)$ by $s^\perp = \pi_{s(1)^\perp}$ for all $s \in \mathbf{Lin}(X)$. Then $(\mathbf{Lin}(X), \circ, \mathrm{id})$ is a Foulis semigroup with respect to taking adjoints * and $^\perp$.

Foulis quantales

The Foulis quantales we introduce here can be characterized precisely as unital involutive quantales that additionally exhibit the structural properties of Foulis semigroups.

Definition 13

A ***Foulis quantale*** is a unital involutive quantale Q together with an endomap ${}^{\perp}\colon Q \to Q$ such that Q is a Foulis semigroup with respect to involution * and operation ${}^{\perp}$.
We will call elements of $[\,Q\,] = \{u^{\perp} \mid u \in Q\}$ ***Sasaki projections***.
A homomorphism of Foulis quantales is a map $h\colon Q_1 \to Q_2$ between Foulis quantales that preserves arbitrary joins, multiplication, unit, involution, and ${}^{\perp}$. In particular, h maps Sasaki projections to Sasaki projections.

Foulis quantale $(\mathbf{Lin}(X), \bigsqcup, \circ, {}^*, {}^\perp, \mathrm{id})$

Proposition 14

$(\mathbf{Lin}(X), \bigsqcup, \circ, {}^*, {}^\perp, \mathrm{id})$ *is a Foulis quantale.*

Theorem 15

Let Q be a Foulis quantale. Then, for all $t, r \in Q$ and $k \in [\,Q\,]$,

$$r \perp t \Longleftrightarrow r^* \cdot t = 0 \Longleftrightarrow t = r^\perp \cdot t \Longleftrightarrow t \le r^\perp \qquad (*)$$

$$t \le r \quad \Longrightarrow \quad r^\perp \le t^\perp \quad \text{and} \quad k^{\perp\perp} = k, \qquad (**)$$

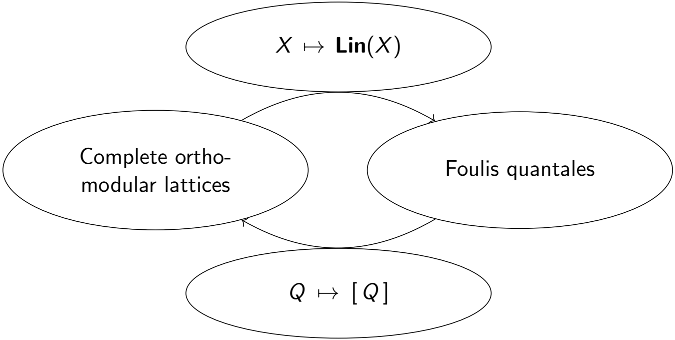
$$t \le r^\perp \quad \Longleftrightarrow \quad r \le t^\perp. \qquad (***)$$

and the subset $[\,Q\,]$ is an orthomodular lattice with the following structure.

Foulis quantale Q and its Sasaki projections $[\,Q\,]$

Order	$k_1 \leq k_2$	$\Leftrightarrow$	$k_1 = k_2 \cdot k_1$
Top	1	$=$	$0^{\perp}$
Orthocomplement	$k^{\perp}$	$=$	$k^{\perp}$
Finite binary meet	$k_1 \wedge k_2$	$=$	$\big(k_1 \cdot \big(k_1 \cdot k_2^{\perp}\big)^{\perp}\big)^{\perp\perp}$
Arbitrary join	$\bigvee X$	$=$	$(\bigsqcup X)^{\perp\perp}$.

Foulis quantales and complete OMLs



- Here $\mathbf{Lin}(X)$ is the Foulis quantale of linear maps on a complete OML X, and

$$[Q] = \{[t] \mid t \in Q\} \subseteq Q,$$

is the complete OML constructed from Foulis quantale Q.

Complete OMLs are quantale modules

The following statement says that a complete orthomodular lattice X can be acted upon from the left by its linear transformations and from the right by a 2-element chain, giving it two different but compatible ways of being transformed or modified.

Proposition 16

Let X be a complete orthomodular lattice. Then X is a left $\mathbf{Lin}(X)$-module and also a right $\mathbf{2}$-module.

Theorem 17

Let Q be a Foulis quantale. Then $[\,Q\,]$ is a left Q-module with action $\bullet$ defined as $u \bullet k = (u \cdot k)^{\perp\perp}$ for all $u \in Q$ and $k \in [\,Q\,]$ and also a right $\mathbf{2}$-module.

Sasaki actions

Definition 18

Let Q be a Foulis quantale and $u \in Q$. The map $\sigma_u : [\,Q\,] \to [\,Q\,]$, $y \mapsto u \bullet y$ is called the *Sasaki action* to $u \in Q$.

Evidently, $\sigma_u \in \mathbf{Lin}([\,Q\,])$. Moreover, if $u \in [\,Q\,]$ then σ_u is self-adjoint linear, idempotent and $\operatorname{im} \sigma_u = {\downarrow} u$ in $[\,Q\,]$.

The following theorem establishes a canonical correspondence between elements of a Foulis quantale and linear transformations acting on its Sasaki projections, illuminating the structural relationship between these components.

Sasaki actions

Theorem 19

Let Q be a Foulis quantale. Then there is a natural homomorphism $h\colon Q \to \mathbf{Lin}([\,Q\,])$ of Foulis quantales such that $h(u) = \sigma_u$ for all $u \in Q$. Moreover, we have a factorization

$$\begin{array}{ccc} Q_\alpha & \xrightarrow{\;h|_{Q_\alpha}\;} & \boldsymbol{Lin}([\,Q\,]) \\ {\scriptstyle j_\alpha}\uparrow & \nearrow{\scriptstyle h} & \\ Q & & \end{array}$$

Here $u \sim_\alpha v$ iff $\sigma_u = \sigma_v$, $j_\alpha(u) = \bigsqcup\{v \in Q \mid u \sim_\alpha v\} \sim_\alpha u$, $Q_\alpha = \{j_\alpha(u) \mid u \in Q\}$ is a Foulis quantale with induced operations such that $[\,Q\,] = [\,Q_\alpha\,]$, j_α is a surjective homomorphism of Foulis quantales, and $h|_{Q_\alpha}$ is an embedding of Foulis quantales.

Final remarks

This presentation introduced a novel method for structuring complete orthomodular lattices as dagger categories.

Leveraging this framework, we established a connection between complete orthomodular lattices and quantales, demonstrating that every complete orthomodular lattice can be represented as a quantale module over a Foulis quantale.

Conversely, we show that each Foulis quantale generates a complete orthomodular lattice, which is also a quantale module over the original Foulis quantale.

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Thank you for your attention!