Decompositions of Posets with least elements

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Blansko, 7-12 September 2025

Literature Review and Motivation

- It is a classical result of lattice theory that there is a bijective correspondence between direct (product) decompositions of a lattice L with a least element 0 into two components and pairs of complementary neutral elements (I, J) of the lattice of ideals $Id(L)$ of L (see [2], Theorem III.4.2).
- An analogous result for Scott-domains has been proved in [4] (Theorem 17).
- Scott-domains and their decompositions are an important tools in the theory of generalized relational databases.

Literature Review

Lattices	Scott-Domains	Posets
• Ideal	• Stable Subdomain	• Finitely stable subposet • (Strongly) stable subposet • (Week) schem

Aim of this talk

Let $(P, \leq_P)$ be a poset.
Purpose: To introduce and characterize all pairs (A, B) of those subsets of P for which each element $p \in P$ has a unique representation of the form $p = a \vee b$, where $a \in A$ and $b \in B$.

Direct product of posets

Definition

The direct product of a non-empty family $\{(P_i, \leq_{P_i}) : \ i \in I\}$ of posets is a pair $(\prod_{i \in I} P_i, \leq_{prod})$ such that:

- $\prod_{i \in I} P_i$ is the direct product of sets $\{P_i\}_{i \in I}$ and;
- $$(x_i)_{i \in I} \leq_{prod} (y_i)_{i \in I} \text{ if and only if } x_i \leq_{P_i} y_i \text{ for all } i \in I.$$

Neutral elements

Definition

An element I of a lattice L is called neutral if, for all $x, y \in L$,

$$(I \wedge_L x) \vee_L (x \wedge_L y) \vee_L (y \wedge_L I) = (I \vee_L x) \wedge_L (x \vee_L y) \wedge_L (y \vee_L I).$$

Order ideals and Ideals

Definition

Let $(P, \leq_P)$ be a poset.

(a) A non-empty subset I of P is called an order ideal of P if, whenever, $x \in I, y \in P$ and $y \leq_P x$, we have $y \in I$.

(b) An order ideal I of P is called an ideal of P if I is a upward directed set, if for every $a, b \in I$ there exists $c \in I$ such that $a, b \leq_P c$.

(c) The set $\downarrow p := \{a \in P \mid a \leq_P p\}$ is an ideal of P for each $p \in P$. Ideals of this kind are called principal.

Lattices of order ideals and ideals

For each poset P, let $(\mathcal{OI}(P), \subseteq)$ and $(\mathcal{I}(P), \subseteq)$ denote the posets of all order ideals and ideals of P, respectively, partially ordered by inclusion $\subseteq$.

- The poset $\mathcal{OI}(P)$ of all order ideals of a poset P is a lattice (respectively, a complete lattice) if and only if P is a downwards directed poset (respectively, P has a least element).
- The poset $\mathcal{I}(P)$ of all ideals of a poset P is a lattice (respectively, a complete lattice) if and only if P is a downwards directed join-semilattice (respectively, P is a join-semilattice with a least element).

A variety of stable sub-posets

Definition

Let $(P, \leq_P)$ be a poset.

(a) A non-empty set $A \subseteq P$ is called a finitely stable subposet of P if A is an order ideal which is closed under all existing finite suprema.

(b) A non-empty set $A \subseteq P$ is called a stable subposet of P if A is an order ideal which is closed under all existing suprema.

A variety of stable sub-posets

Definition

(c) An order ideal A of P is a strongly stable subposet of P if, for all $p \in P$, $\downarrow p \cap A$ has a greatest element.
- In particular, we can define the map $\pi_A \colon P \to A$, $\pi_A(p) := \bigvee \downarrow p \cap A$, for each $p \in P$.

(d) Let $\mathcal{S}_f(P)$, $\mathcal{S}(P)$ and $\mathcal{S}_s(P)$ denote the families of all finitely stable, stable and strongly stable subposets of P, respectively.

Some facts on these special subposets

Let $(P, \leq_P)$ be a poset and $p \in P$. Then

(a) each ideal I of a poset P is a finitely stable subposet in P, because each finite set $F \subseteq I$ has an upper bound in I ($\mathcal{I}(P) \subseteq \mathcal{S}_f(P)$);

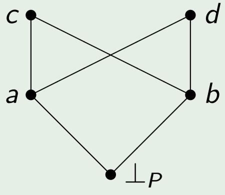
(b) $\mathcal{S}_s(P) \subseteq \mathcal{S}(P)$;

(c) $\mathcal{S}(P) \subseteq \mathcal{S}_f(P)$.

Finitely stable subposets need not be ideals

Example

The pair a, b does not have a supremum. Thus $A = \{\bot_P, a, b\}$ is a (finitely) stable subposet of P. However, A is not an ideal of P, because it is not a directed set.

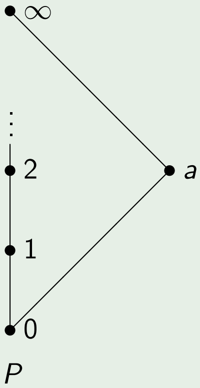


P

Finitely stable subposets (ideals) are not stable subposets

Example

Let $P = \mathbb{N} \cup \{\infty\}$. Then $\mathbb{N} = \{0, 1, 2, \ldots\}$ is finitely stable but not stable, since $\infty = \bigvee \mathbb{N} \notin \mathbb{N}$.



Stable subposets are not necessarily stongly stable

Take the three-element poset $P = \{a, b, \top_P\}$ with the greatest element $\top_P$ such that elements a and b are not comparable, i.e., the set $\{a, b\}$ form two-element antichain. Then $\mathcal{OI}(P) = \{\downarrow_P a,\ \downarrow_P b, \{a, b\}, P\}$. Next, $\downarrow_P a \cap \downarrow_P b = \emptyset$ which implies that $\downarrow_P a$ and $\downarrow_P b$ are not strongly stable subposets of P. Moreover, $\downarrow_P \top_P \cap \{a, b\} = \{a, b\}$ does not have a greatest element.

- Thus $\mathcal{S}_s(P) = \{P\}$.
- $\mathcal{S}_f(P) = \mathcal{S}(P) = \mathcal{I}(P) = \{\downarrow_P a,\ \downarrow_P b, P\}$.

Strongly stable subposets-stable subposets

Theorem

Let $(P, \leq_P)$ be a poset. Then a set $A \subseteq P$ is a strongly stable subposet of P if and only if A is a stable subposet of P and the supremum $\bigvee_P(\downarrow p \cap A)$ exists for each $p \in P$.

Projections

Definition

Let $(P, \leq_P)$ be a poset. Then a map $\pi\colon P \longrightarrow P$ is called a projection if:

- π is monotone (i.e., $p_1 \leq_P p_2$ implies $\pi(p_1) \leq_P \pi(p_2)$);
- π is idempotent (i.e., $\pi(\pi(p)) = \pi(p)$ for all $p \in P$);
- $\pi(p) \leq_P p$, for all $p \in P$.

Characterization of projections

Theorem

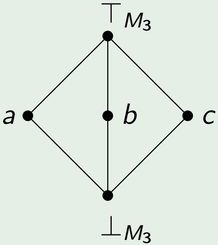
Let $(P, \leq_P)$ be a poset. Then an order ideal A of P is a strongly stable subsposet of P if and only if there is a projection $\pi\colon P \longrightarrow P$ with $\pi(P) = A$.

Projections do not preserve existing suprema

Example

Take the five element lattice M_3 and let $A = \{\bot_{M_3}, a\}$. The subposet A is a strongly stable subposet of M_3 while π_A does not preserve suprema. In fact,

$$\pi_A(b \vee_{M_3} c) = \pi_A(\top_{M_3}) = a \neq \bot_{M_3} = \bot_{M_3} \vee_{M_3} \bot_{M_3} = \pi_A(b) \vee_{M_3} \pi_A(c).$$



M_3

Schems

Definition

Let $(P, \leq_P)$ be a poset and A a strongly stable subposet of P. Then

(a) A is called a scheme if the projection $\pi_A \colon P \longrightarrow A$ preserves all existing suprema, i.e., for each subset $X \subseteq P$, if the supremum $\bigvee_P X$ exists, then the supremum $\bigvee_P \pi_A(X)$ exists and $\bigvee_P \pi_A(X) = \pi_A(\bigvee_P X)$.

(b) A is called a weak scheme if the projection $\pi_A \colon P \longrightarrow A$ preserves all existing finite suprema.

Lattice of stable subposets

Let $(P, \leq_P)$ be a poset with a least element $\perp_P$.

(a) $\mathcal{S}(P)$ (respectively, $\mathcal{S}_f(P)$) partially ordered by inclusion is a complete lattice in which the meet of an arbitrary non-empty family of stable (respectively, finitely stable) subposets is given by its intersection. Moreover, $\{\perp_P\}$ is the least and P is the greatest element of this lattice.

(c) For each two strongly stable subposets A and B of P, the intersection $A \cap B$ is also a strongly stable subposet, and moreover, $\pi_{A\cap B} = \pi_A \circ \pi_B = \pi_B \circ \pi_A$.

(d) For each two schemes (respectively, weak schemes) A and B of P, the intersection $A \cap B$ is also a scheme (respectively, a weak scheme).

(e) $Ss(P)$, $C_w(P)$ and $C(P)$ partially ordered by inclusion are meet-semilattices

Three sorts of decomposition

Definition

Let $(P, \leq_P)$ be a poset with a least element $\perp_P$. Then

(a) A pair (A, B) of strongly stable subposets of P is called a quasi-general decomposition of P (into two factors) if each $p \in P$ has a unique representation as $p = a \vee_P b$ such that $a \in A$ and $b \in B$.

(b) A quasi-general decomposition (A, B) of P is called a weak general decomposition of P (into two factors) if A and B are weak schemes of P.

(c) A quasi-general decomposition (A, B) of P is called a general decomposition of P (into two factors) if A and B are schemes of P.

Properties of quasi-general decomposition

Theorem

Let $(P, \leq_P)$ be a poset with a least element $\bot_P$, and let (A, B) be a quasi-general decomposition of P. Then the following conditions are satisfied.

(a) $A \cap B = \{\bot_P\}$, *i.e.,* $A \wedge_{S(P)} B = \{\bot_P\}$ *and* $A \wedge_{S_f(P)} B = \{\bot_P\}$.

(b) $\{a \vee_P b \colon\ a \in A,\ b \in B$ *and the join* $a \vee_P b$ *exists* $\} = P$. *In particular,* $A \vee_{S(P)} B = P$ *and* $A \vee_{S_f(P)} B = P$.

(c) *For each* $a \in A$ *and* $b \in B$, $\pi_A(b) = \bot_P$ *and* $\pi_B(a) = \bot_P$.

(d) *For every* $a \in A$ *and* $b \in B$, *if the supremum* $a \vee_P b$ *exists, then* $\pi_A(a \vee_P b) = a$ *and* $\pi_B(a \vee_P b) = b$.

(e) *For each* $p \in P$, *the supremum* $\pi_A(p) \vee_P \pi_B(p)$ *exists and equals* p. *In particular,* $\pi_A(p) \vee_P \pi_B(p)$ *is the unique representation of* p.

Uniqueness of representation of element of a poset via singleton-set intersection

Theorem

Let $(P, \leq_P)$ be a poset with a least element $\perp_P$. Then the pair (A, B) of strongly stable subposets of P is a general (respectively, weak general) decomposition of P if and only if A and B are schemes (respectively, weak schemes) of P such that $A \cap B = \{\perp_P\}$, and each $p \in P$ has a representation as $p = a \vee_P b$ where $a \in A$ and $b \in B$.

Necessary and sufficient condition for (week/quasi-) general decomposition to be the direct decomposition

Theorem

Let $(P, \leq_P)$ be a poset with a least element $\bot_P$. Then the following conditions are equivalent:

(a) *P is isomorphic to the direct product $A \times B$ of posets A and B,*
(b) *the pair (A, B) is (up to isomorphism) a general decomposition of P such that the supremum $a \vee_P b$ exists for all $a \in A$ and $b \in B$.*
(c) *the pair (A, B) is (up to isomorphism) a weak general decomposition of P such that the supremum $a \vee_P b$ exists for all $a \in A$ and $b \in B$.*
(d) *the pair (A, B) is (up to isomorphism) a quasi-general decomposition of P such that the supremum $a \vee_P b$ exists for all $a \in A$ and $b \in B$.*

Bijective correspondence (week-) general decompositions and neutral elements of ($\mathcal{S}_f(P)$) $\mathcal{S}(P)$

Theorem

Let $(P, \leq_P)$ be a poset with a least element $\bot_P$. Then a pair (A, B) is a general decomposition (respectively, a weak general decomposition) of P if and only if the following conditions hold:

(a) *A and B are strongly stable subposets of P,*

(b) *A and B are neutral elements of the lattice $S(P)$ of all stable subposets of P (respectively, of the lattice $S_f(P)$ of all finitely stable subposets of P) complementing each other.*

Main result

Theorem

Let $(P, \leq_P)$ be a poset with a least element $\perp_P$. Then the following conditions are equivalent:

(a) *P is isomorphic to the direct product $A \times B$ of two posets.*

(b) *A and B are (up to isomorphism) strongly stable subposets of P, which are also neutral elements of $S(P)$ complementing each other, and the supremum $a \vee_P b$ exists for all $a \in A$ and $b \in B$.*

(c) *A and B are (up to isomorphism) strongly stable subposets of P, which are also neutral elements of $S_f(P)$ complementing each other, and the supremum $a \vee_P b$ exists for all $a \in A$ and $b \in B$.*

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Thank you!