Minimal operations over permutation groups

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SSAOS, September 8th, 2025

POCOCOP ERC Synergy Grant No. 101071674. Views and opinions expressed are those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Outline

Clones

Definition 1 (Clone)

Let B be a (possibly infinite) set.
Let $\mathcal{O}^{(n)} = B^{B^n}$ be the set of functions $f : B^n \to B$, and
$\mathcal{O} := \bigcup_{n\in\mathbb{N}} \mathcal{O}^{(n)}$.
We call $\mathcal{C} \subseteq \mathcal{O}$ a (function) **clone** over B if

- $\mathcal{C}$ contains all projections;
- $\mathcal{C}$ is closed under composition.

For $\mathcal{S} \subseteq \mathcal{O}$, $\langle \mathcal{S} \rangle$ is the smallest clone containing $\mathcal{S}$.

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Closed clones

Interested in clones which are **closed** in the **pointwise convergence topology**:[1] For $\mathcal{S} \subseteq \mathcal{O}$,
$f \in \overline{\mathcal{S}} \Leftrightarrow$ for all $A \subseteq B$ finite there is $g \in \mathcal{S}$ such that $g_{\restriction A} = f_{\restriction A}$.

For B finite, topology trivialises (i.e. closed clone=clone).

$\overline{\langle \mathcal{S} \rangle}$ denotes the smallest closed clone containing $\mathcal{S}$.

There is a correspondence between:

- closed clones on B;
- polymorphism clones of relational structures on B.

▸ Definition of polymorphism clone

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Monoidal intervals

Let $\mathcal{T}$ be a **transformation monoid** on B
(i.e. unary operations containing Id, and closed under composition).

Closed clones whose unary operations are $\overline{\mathcal{T}}$ form an interval in the lattice of closed clones on B, known as the **monoidal interval of $\mathcal{T}$**.

Studying the structure and size of monoidal intervals has a long history in universal algebra:

- for $\mathcal{O}_B^{(1)}$ (Burle 1967);
- for $G \curvearrowright B$ a permutation group (Pálfy and Szendrei 1982; Kearnes and Szendrei 2001) (with focus on collapse);
- for other monoids (Krokhin 1995);
- over infinite sets (Pinsker 2008).

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Minimal operations

What is the minimal amount of structure in a clone containing $\mathcal{T}$?

Definition 2 (Minimal clone)

Let $\mathcal{D} \supsetneq \mathcal{C}$ be closed clones.
$\mathcal{D}$ is **minimal above** $\mathcal{C}$ if there is no closed clone $\mathcal{E}$ such that $\mathcal{C} \subsetneq \mathcal{E} \subsetneq \mathcal{D}$.

Definition 3 (almost minimal and minimal operations)

The k-ary operation $f \in \mathcal{O} \setminus \mathcal{C}$ is **almost minimal above** $\mathcal{C}$ if for each $r < k$,

$$\overline{\langle \mathcal{C} \cup \{f\} \rangle} \cap \mathcal{O}^{(r)} = \mathcal{C} \cap \mathcal{O}^{(r)}.$$

If f is almost minimal above $\mathcal{C}$ and $\overline{\langle \mathcal{C} \cup \{f\} \rangle}$ is minimal above $\mathcal{C}$, then f is **minimal above** $\mathcal{C}$.

Minimal operations

What is the minimal amount of structure in a clone containing $\mathcal{T}$?

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Basic facts on minimality and almost minimality

- $\mathcal{D}$ is minimal above $\mathcal{C}$ if and only if $\mathcal{D} = \overline{\langle \mathcal{C} \cup \{f\} \rangle}$ for f minimal above $\mathcal{C}$;
- minimal elements in the interval of $\mathcal{T}$ above $\overline{\langle \mathcal{T} \rangle}$ correspond to minimal clones above $\overline{\langle \mathcal{T} \rangle}$ which are not essentially unary[2];
- ALWAYS, if $\mathcal{E} \supsetneq \mathcal{C}$, there is $f \in \mathcal{E} \setminus \mathcal{C}$ almost minimal above $\mathcal{C}$;

We will study minimal operations above $\overline{\langle G \rangle}$ for $G \curvearrowright B$ a non-trivial permutation group.

[2] f is **essentially unary** if it depends on only one variable. Otherwise, it is **essential**.

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Oligomorphic permutation groups

We are particularly interested in oligomorphic permutation groups:

Definition 4 (Oligomorphicity, ω-categoricity)

B countably infinite. $G \curvearrowright B$ is **oligomorphic** if $G \curvearrowright B^n$ has finitely many orbits for each $n \in \mathbb{N}$;
A first-order structure $\mathbb{B}$ is **ω-categorical** if $\mathrm{Aut}(\mathbb{B}) \curvearrowright B$ is oligomorphic.

Examples of ω-categorical structures:

- $(\mathbb{N}, =)$;
- $(\mathbb{Q}, <)$;
- The Random graph;
- Countable vector spaces over finite fields.

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Fact 5 (Existence of minimal operations)

Let $\mathcal{C} \subsetneq \mathcal{D}$ be closed function clones. Suppose either

- *B is finite; or*
- *$\mathcal{C} = \overline{\langle \mathrm{Aut}(\mathbb{B}) \rangle}$ for $\mathbb{B}$ ω-categorical in a finite relational language.*

Then, there is $\mathcal{E} \subseteq \mathcal{D}$ which is minimal above $\mathcal{C}$.

Note: in general, this can fail when B is infinite.

Rosenberg's five types theorem

Theorem 6 (Five types theorem, Rosenberg 1986)

Let B be finite and f be minimal above $\langle \mathrm{Id} \rangle$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary majority operation;*
4. *a minority of the form $x + y + z$ in some Boolean group $(B, +)$;*
5. *a k-ary semiprojection for some $k \geq 3$.*

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Ternary majority: an operation $m : B^3 \to B$ such that

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx m(x, x, x) \approx x;$$

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A group is **Boolean** if every non-identity element has order 2.
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Semiprojection: $f : B^k \to B$ such that there is an $i \in \{1, \ldots, k\}$ such that whenever $(a_1, \ldots, a_k)$ is a non-injective tuple from B,

$$f(a_1, \ldots, a_k) = a_i.$$

The oligomorphic case

Bodirsky and Chen 2007 classify minimal operations above oligomorphic permutation groups.

Theorem 7 (Oligomorphic case, Bodirsky and Chen 2007)

Let $G \curvearrowright B$ be an oligomorphic permutation group. Let f be minimal above $\overline{\langle G \rangle}$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary quasi-majority operation;*
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Ternary quasi-majority:[3] an operation $m : \mathbb{B}^3 \to \mathbb{B}$ such that

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx m(x, x, x) \cancel{\approx x};$$

[3] "quasi":= we don't ask that $f(x, \ldots, x) \approx x$.

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Quasi-semiprojection: $f : B^k \to B$ such that there is an $i \in \{1, \dots, k\}$ and $g \in \overline{G}$ such that whenever $(a_1, \dots, a_k)$ is a non-injective tuple,

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Note: No minority type!

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We obtain better results even in this case!

The Main Theorem

Theorem 8 (Minimal operation, MP 2024)

Let $G \curvearrowright B$ be non-trivial with s many orbits (s possibly infinite). Let f be a minimal operation above $\overline{\langle G \rangle}$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary quasi-minority operation of the form $\alpha\mathfrak{q}$ for $\alpha \in G$, where*
   - *G is a Boolean group acting freely on B with $s = 2^n$ or infinite;*
   - *the operation $\mathfrak{q}$ is a G-invariant Boolean Steiner 3-quasigroup.*
4. *f is a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

- type 3 strengthens quasi-minority. Essentially never occurs;
- type 4 strengthens quasi-semiprojection. Arity bounded by orbits;
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In the oligomorphic case we improve on Bodirsky and Chen 2007:

- We reduced from four to three types ($G \curvearrowright B$ is not free);
- Stronger characterisation of the quasi-semiprojection case.

G-invariant Boolean Steiner 3-quasigroups

Definition 9

A G-invariant Boolean **Steiner 3-quasigroup**[3] is a symmetric ternary minority operation $\mathfrak{q}$ satisfying:

$$\mathfrak{q}(x, y, \mathfrak{q}(x, y, z)) \approx z \; ; \qquad \text{(SQS)}$$

$$\mathfrak{q}(x, y, \mathfrak{q}(z, y, w)) \approx \mathfrak{q}(x, z, w) \; ; \qquad \text{(Bool)}$$

$$\text{for all } \alpha, \beta, \gamma \in G, \mathfrak{q}(\alpha x, \beta y, \gamma z) \approx \alpha\beta\gamma\mathfrak{q}(x, y, z). \qquad \text{(Inv)}$$

[3] Studied in universal algebra (Quackenbush 1975; Ganter and Werner 1975) and design theory (Lindner and Rosa 1978).

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$$\text{for all } \alpha, \beta, \gamma \in G, \mathfrak{q}(\alpha x, \beta y, \gamma z) \approx \alpha\beta\gamma\mathfrak{q}(x, y, z). \tag{Inv}$$

(SQS) yields that $\mathfrak{q}(x_1, x_2, x_3) = x_4 \wedge \bigwedge_{i<j\leq 4} x_i \neq x_j$ is a **Steiner quadruple system** on B: a 4-hypergraph on B such that every three vertices are in a unique 4-hyperedge.

Steiner 3-quasigroups correspond to Steiner quadruple systems on B.

[3] Studied in universal algebra (Quackenbush 1975; Ganter and Werner 1975) and design theory (Lindner and Rosa 1978).

G-invariant Boolean Steiner 3-quasigroups

Definition 9

A G-invariant **Boolean Steiner $\mathbf{3}$-quasigroup**[3] is a symmetric ternary minority operation $\mathfrak{q}$ satisfying:

$$\mathfrak{q}(x, y, \mathfrak{q}(x, y, z)) \approx z \; ; \qquad \text{(SQS)}$$

$$\mathfrak{q}(x, y, \mathfrak{q}(z, y, w)) \approx \mathfrak{q}(x, z, w) \; ; \qquad \text{(Bool)}$$

for all $\alpha, \beta, \gamma \in G, \mathfrak{q}(\alpha x, \beta y, \gamma z) \approx \alpha\beta\gamma\mathfrak{q}(x, y, z)$. (Inv)

Correspond to $x + y + z$ on a Boolean group on B.

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$\mathfrak{q}(x,y,z)$ also induces a Boolean Steiner 3-quasigroup on the G-orbits.

Description of minimal minorities

Theorem 10 (Description of minimal minorities, MP 2024)

Let $G \curvearrowright B$ be a Boolean group acting freely on B with s many orbits.

- *All G-invariant Boolean Steiner 3-quasigroups are minimal above $\overline{\langle G \rangle}$;*
- *There are G-invariant Boolean Steiner 3-quasigroups if and only if $s = 2^n$ for some $n \in \mathbb{N}$, or is infinite;*
- *For $s = 2^n$, the number of G-invariant Boolean Steiner 3-quasigroups is 1 for $n = 0$, and for $n \geq 1$ it is*

$$\frac{(2^n - 1)!|G|^{(2^n - n - 1)}}{\prod_{k=0}^{n-1}(2^n - 2^k)}.$$

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$$\frac{(2^n - 1)!|G|^{(2^n - n - 1)}}{\prod_{k=0}^{n-1}(2^n - 2^k)} .$$

Counting G-invariant Boolean Steiner 3-quasigroups requires solving a small problem in projective geometry.

Orbit-semiprojections

f is an **orbit-semiprojection** if there is an $i \in \{1, \ldots, k\}$ and $g \in \overline{G}$ such that whenever at least two of the a_j lie in the same orbit,

$$f(a_1, \ldots, a_k) = g(a_i).$$

- Almost minimal k-ary orbit-semiprojections exist for all $2 \leq k \leq s$;
- Pálfy 1986: k-ary semiprojections minimal above $\langle \mathrm{Id} \rangle$ exist for each $2 \leq k \leq |B|$.

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Theorem 11 (Pálfy's theorem for orbit-semiprojections, MP 2024)

Let $G \curvearrowright B$ with s-many orbits be finite or oligomorphic. Then, for all $2 \leq k \leq s$, there is a k-ary orbit-semiprojection minimal above $\overline{\langle G \rangle}$.

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When G is finite (non-trivial), taking $b \in B$ in an orbit of size > 1, let

$$f(a_1, \ldots, a_k) := \begin{cases} b \text{ if } a_1 \sim b \text{ and } a_1, \ldots, a_k \text{ are all in distinct orbits;} \\ a_1 \text{ otherwise.} \end{cases}$$

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When $G \curvearrowright B$ is oligomorphic, taking $b \in B$ in an orbit of size > 1, letting γ belong to a minimal closed $(\overline{G}_b, \overline{G})$-biact,[3]

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[3] A $(\overline{G}_b, \overline{G})$-biact is a set $\mathcal{I} \subseteq \overline{G}$ such that $\overline{G}_b \mathcal{I} \overline{G} \subseteq \mathcal{I}$

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Let $G \curvearrowright B$ with s-many orbits be finite or oligomorphic. Then, for all $2 \leq k \leq s$, there is a k-ary orbit-semiprojection minimal above $\overline{\langle G \rangle}$.

- always holds for minimality in the lattice of *all* clones (rather than closed clones);
- **Problem:** Find $G \curvearrowright B$ with three orbits such that there is no ternary orbit-semiprojection minimal above $\overline{\langle G \rangle}$.

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Methods: almost minimality

To classify minimal operations, we first classify almost minimal operations above $\overline{\langle G \rangle}$:

- Start from a weak version of Rosenberg's Theorem for almost minimal operations above a monoid without constant operations (implicit in Bodirsky and Chen 2007);
- Show that certain behaviours cannot be witnessed by almost minimal operations since they give rise to non-injective unary operations.

This splits into three cases:

- $G \curvearrowright B$ not a Boolean group acting freely on B;
- $G \curvearrowright B$ a Boolean group acting freely on B with $|G| > 2$;
- $\mathbb{Z}_2 \curvearrowright B$ acting freely.

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Case1: The three types

Lemma 12 (Three types lemma, MP 2024)

Let $G \curvearrowright B$ with s-many orbits be such that G is not a Boolean group acting freely on B. Let f be almost minimal above $\overline{\langle G \rangle}$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

Case 2: the Boolean case

Lemma 13 (Boolean case, MP 2024)

Let $G \curvearrowright B$ be a Boolean group acting freely on B with s-many orbits and $|G| > 2$. Let f be an almost minimal operation above $\langle G \rangle$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a G-quasi-minority;*
4. *a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

A ***G*-quasi-minority** is a ternary operation such that for all $\beta \in G$,

$$\mathfrak{m}(y, x, \beta x) \approx \mathfrak{m}(x, \beta x, y) \approx \mathfrak{m}(x, y, \beta x) \approx \mathfrak{m}(\beta y, \beta y, \beta y).$$

Case 3: the $\mathbb{Z}_2$ case

Lemma 14 ($\mathbb{Z}_2$ case, MP 2024)

Let $\mathbb{Z}_2$ act freely on B with s-many orbits. Let f be an almost minimal operation above $\langle\mathbb{Z}_2\rangle$. Then, f is one of

1. *a unary operation;*
2. *a G-quasi-minority;*
3. *an odd majority;*
4. *an odd Malcev, up to permuting variables;*
5. *a k-ary orbit-semiprojection for $2 \leq k \leq s$.*

An **odd majority** m is a ternary quasi-majority such that for $\gamma \neq \mathrm{Id}$ in $\mathbb{Z}_2$,

$$m(y, x, \gamma x) \approx m(x, \gamma x, y) \approx m(x, y, \gamma x) \approx m(y, y, y).$$

An **odd Malcev** is such that $M(x, \gamma y, z)$ is an odd majority.
Odd majorities and odd Malcev cannot be minimal!

Motivation from CSPs

$\tau :=$ finite relational language.
$\mathbb{B} :=$ a fixed τ-structure.

Definition 15 ($\mathrm{CSP}(\mathbb{B})$)

$\mathrm{CSP}(\mathbb{B})$ is the following computational problem:

- **INPUT**: A **finite** τ-structure $\mathbb{A}$;
- **OUTPUT**: Is there a homomorphism $\mathbb{A} \to \mathbb{B}$?

We study $\mathrm{CSP}(\mathbb{B})$ for $\mathbb{B}$ finite or ω-categorical.

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The algebraic approach to CSPs

Algebraic approach to CSPs:

The polymorphisms of $\mathbb{B}$ capture the computational complexity of $\mathrm{CSP}(\mathbb{B})$.

Highly successful in the finite setting:

Theorem 16 (Bulatov 2017; Zhuk 2017)

Let $\mathbb{B}$ be finite. Then:

- *EITHER $\mathbb{B}$ has a Siggers polymorphism.*[3]
  In this case, $\mathrm{CSP}(\mathbb{B})$ is in P;
- *OR $\mathbb{B}$ "pp-constructs" EVERYTHING (i.e., all finite structures)*
  In this case, $\mathrm{CSP}(\mathbb{B})$ is NP-complete.

[3] A polymorphism $s : \mathbb{B}^6 \to \mathbb{B}$ such that

$$s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y).$$

The algebraic approach to CSPs

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The polymorphisms of $\mathbb{B}$ capture the computational complexity of $\mathrm{CSP}(\mathbb{B})$.

Often successful for $\mathbb{B}$ ω-categorical: complexity dichotomies for CSPs of structures first-order definable in:

- $(\mathbb{Q}, <)$ (Bodirsky and Kára 2010);
- homogeneous graphs (Bodirsky, Martin, Pinsker, and Pongrácz 2019);
- countable unary structures (Bodirsky and Mottet 2018);

  ⋮

Bodirsky-Pinsker conjecture: CSPs of a large class of ω-categorical structures[3] satisfy a complexity dichotomy analogous to the finite-domain one.

[3] First-order reducts of finitely bounded homogeneous structures.

Understanding low arity polymorphisms

Question 1

What is the minimal amount of structure in $\mathrm{Pol}(\mathbb{B})$ when $\mathrm{CSP}(\mathbb{B})$ is not NP-hard (due to pp-constructing EVERYTHING)?

- Sufficient to consider case of a (model complete) **core**: $\overline{\mathrm{Aut}(\mathbb{B})} = \mathrm{End}(\mathbb{B})$;
- We can assume $\mathrm{Pol}(\mathbb{B})$ is **essential**: if $\mathrm{Pol}(\mathbb{B})$ is NOT essential, then $\mathbb{B}$ pp-interprets EVERYTHING;
- **Bottom-up approach to CSPs:** several complexity classifications identify the behaviours of low arity essential polymorphisms.

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Binary essential polymorphisms

Complexity classifications of ω-categorical CSPs often show that under tame assumptions $\mathbb{B}$ has a binary essential polymorphism!

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(Bodirsky and Kára 2008):

Lemma 5. *Every essentially at least binary operation together with all permutations locally generates a binary operation that depends on both arguments.*

(Bodirsky and Kára 2010)

Lemma 10. *Let Γ be a relational structure and let R be a k-ary relation that is a union of l orbits of k-tuples of $Aut(\Gamma)$. If R is violated by a polymorphism g of Γ of arity $m \geq l$, then R is also violated by an l-ary polymorphism of Γ.*

(Bodirsky and Pinsker 2014):

LEMMA 40: *Let $f\colon V^k \to V$ be an essential operation. Then f generates a binary essential operation.*

(Bodirsky 2021; Mottet and Pinsker 2024):

LEMMA 6.1.29. *Let $\mathscr{C}$ be a clone with an essential operation that contains a permutation group $\mathscr{G}$ with the orbital extension property. Then $\mathscr{C}$ must also contain a binary essential operation.*

PROPOSITION 23. *Let $\mathbb{A}$ be a first-order reduct of a homogeneous structure $\mathbb{B}$ such that $\mathbb{B}$ has a free orbit. If $\mathrm{Pol}(\mathbb{A})$ contains an essential function, then it contains a binary essential operation.*

(Mottet, Nagy, and Pinsker 2024)

Lemma 27. *Let $\mathbb{A}$ be a first-order reduct of $\mathbb{H}$ that is a model-complete core. If $\mathrm{Pol}(\mathbb{A})$ does not have a uniformly continuous clone homomorphism to $\mathscr{P}$, then it contains a binary essential operation.*

Also done in:

- Bodirsky, Jonsson, and Van Pham 2017;
- Bodirsky and Mottet 2018;
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ISSUE: These techniques are ad-hoc!

A question of Bodirsky

Question 2 (Question 24 in Bodirsky 2021)

Suppose $\mathbb{B}$ is an ω-categorical (model complete) core and $\mathrm{Pol}(\mathbb{B})$ has an essential polymorphism.
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Answer: No.(Though true if $\mathrm{Aut}(\mathbb{B})$ has ≤ 2 orbits.)
Counterexample: for any oligomorphic $\mathrm{Aut}(\mathbb{B}) \curvearrowright \mathbb{B}$ with s orbits for $s \geq 3$, there is an ω-categorical structure $\mathbb{B}'$ such that

$$\mathrm{Pol}(\mathbb{B}') := \overline{\langle \mathrm{Aut}(\mathbb{B}) \cup \{f\} \rangle},$$

where f is an essential s-ary orbit-semiprojection.

Why easy problems lie above binary operations

Moral answer (for the purposes of CSPs): Yes!

Theorem 16 (Finding binary operations, MP 2024)

Let $G \curvearrowright B$ be such that G is not a Boolean group acting freely on B. Suppose that $\mathcal{C} \cap \mathcal{O}^{(1)} = \overline{G}$, and

($\star$) *$\mathcal{C}$ has no uniformly continuous clone homomorphism*[4] *to $\mathcal{P}_{\{0,1\}}$, the clone of projections on a two-element set.*

Then, $\mathcal{C}$ contains a binary essential operation.

[4] $\xi : \mathcal{C} \to \mathcal{P}_{\{0,1\}}$ is **uniformly continuous** if there is some finite $B' \subseteq B$ such that $f_{\restriction B'} = g_{\restriction B'}$ implies $\xi(f) = \xi(g)$.

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Let $G \curvearrowright B$ be such that G is not a Boolean group acting freely on B. Suppose that $\mathcal{C} \cap \mathcal{O}^{(1)} = \overline{G}$, and

($\star$) *$\mathcal{C}$ has no uniformly continuous clone homomorphism to $\mathcal{P}_{\{0,1\}}$, the clone of projections on a two-element set.*

Then, $\mathcal{C}$ contains a binary essential operation.

Proof idea.

Suppose by contrapositive that $\mathcal{C} \cap \mathcal{O}^{(2)} = \overline{\langle G \rangle} \cap \mathcal{O}^{(2)}$.

Then, all ternary operations in $\mathcal{C}$ are almost minimal.

So $\mathcal{C} \cap \mathcal{O}^{(3)}$=essentially unary operations and orbit-semiprojections.

These will only satisfy trivial identities, which is sufficient to build a uniformly continuous homomorphism to $\mathcal{P}_{\{0,1\}}$. □

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Thank you!

A brief recap:

- We classify minimal (and almost minimal) operations above arbitrary permutation groups;
- We get fewer types than those of Rosenberg's theorem above the trivial group;
- We give general reasons why when $\mathbb{B}$ is an ω-categorical (model complete) core and $\mathrm{CSP}(\mathbb{B})$ is not NP-hard, we can find binary essential polymorphisms;

QR code to preprint:

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Polymorphisms

Let $\mathbb{B}$ be a relational structure.

Definition 17 (Polymorphisms)

$f : \mathbb{B}^n \to \mathbb{B}$ is a **polymorphism** if it **preserves all relations** of $\mathbb{B}$:

$$\begin{pmatrix} a_1^1 \\ \vdots \\ a_k^1 \end{pmatrix}, \ldots, \begin{pmatrix} a_1^n \\ \vdots \\ a_k^n \end{pmatrix} \in R^{\mathbb{B}} \Rightarrow \begin{pmatrix} f(a_1^1, \ldots, a_1^n) \\ \vdots \\ f(a_k^1, \ldots, a_k^n) \end{pmatrix} \in R^{\mathbb{B}}.$$

We call $\mathbf{Pol}(\mathbb{B})$ the set of polymorphisms of $\mathbb{B}$.
The **polymorphism clone of** $\mathbb{B}$.

- Unary polymorphism = endomorphism;
- **Projections** to one coordinate **are always polymorphisms**.

Back to main presentation

pp-interpretations and pp-constructions

A ***pp*-formula** is a first-order formula consisting only of existential quantifiers, conjunctions, and atomic formulas.

Definition 18 (pp-interpretation, pp-construction)

$\mathbb{B}$ ***pp*-interprets** $\mathbb{A}$ if there is partial surjective $h : \mathbb{B}^d \to \mathbb{A}$ such that for every $R \subseteq \mathbb{A}^n$ that is a relation of $\mathbb{A}$ (or $\mathbb{A}$, or equality on $\mathbb{A}$), $h^{-1}(R)$ is defined by a pp-formula in $\mathbb{B}^{nd}$.

$\mathbb{B}$ ***pp*-constructs** $\mathbb{A}$ if it is homomorphically equivalent to a structure that pp-interprets $\mathbb{A}$.