Nonclassical Polyadic Algebras : Soft and Hard

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Summer School on General Algebra and Ordered Sets
11 September 2025

Introduction

J. M. Font explains in [Fon16] that :

- If we associate a class algebras with a logic, then the algebras should have the same similarity types the language of that logic. Thus, the algebraic formulation of predicate logic as cylindric or polyadic algebras is closer to AAL.
- Soft : Evaluate quantifiers as infinite lattice operations Hard : Evaluate quantifiers as unary primitive operations

Question

Can we find connection between the Soft and Hard interpretation of quantifiers ?

Answer: Yes. In [PS95], Don Pigozzi and Antonino Salibra build an equational-like logic for first-order logic through "binding", study the term algebras through polyadic algebras, and prove the completeness theorem with respect to models throguh functional representation theorem.
In [PS93] build the connection between Rasiowa-implicative logic and Polyadic algebra through proving the representation theorems.

Setting

Give two sets I, J with $J \subseteq I$. We call a mapping $\sigma : I \to I$ a transformation of I.

Notation

- The identity transformation is denoted by ι;
- For $\sigma, \tau \in I^I$, $\sigma J \tau$ means that $\sigma(i) = \tau(i)$ for all $i \in J$. Also, we denote $\sigma(I \setminus J)\tau$ as $\sigma J_* \tau$;
- If $\sigma J_* \iota$, we say J *supports* σ.

Classical Polyadic Algebras – 1

Definition 1 ([Hal54])

An (existential) quantifier on a Boolean algebra **A** is a unary operation $\exists : A \to A$ such that

1. $\exists 0 = 0$;
2. $p \leq \exists p$;
3. $\exists(p \wedge \exists q) = \exists p \wedge \exists q$

Classical Polyadic Algebras – 2

Definition 2 ([Hal54])

A (quasi-)polyadic algebra is a quadruple $\langle A, I, S, \exists \rangle$ where **A** is a Boolean algebra, I is a set (for variables), $\exists : P(I) \to A^A$ is a mapping from subsets of I to quantifiers on **A**, and $S : I^I \to Hem(A)$ such that

- $\exists(\emptyset)p = p$ for all $p \in A$
- $\exists(J \cup K) = \exists(J) \circ \exists(K)$ for all subsets J, K of A
- $S(1_I) = 1_A$
- $S(\sigma)(S(\tau)) = S(\sigma\tau)$
- If $J \subset I$ and σ, τ are transformations on I such that $\sigma(I - J) = \tau(I - J)$ then $S(\sigma)\exists(J) = S(\tau)\exists(J)$
- If $J \subseteq I$ and τ is a transformation which is injective on $\tau^{-1}J$, then $\exists(J)S(\tau) = S(\tau)\exists(\tau^{-1}J)$

Classical Polyadic Algebras – 3

Some explanations :

- no substitutions of variables, no corresponding changes to the propositional functions.
- applying substitution $\sigma \circ \tau$ of variables in a propositional function should have the same effect as applying τ first and then applying σ.
- once a variable has been quantified, the replacement of that variable by another one has no further effect.
- once the variable has been replaced by another one, quantification on the replaced variable has no further effect.

Algebraically implicative logics – 1

Definition 3 ([CN21])

A logic L is *algebraically implicative* if there is a binary connective $\to$ (primitive or definable) and a set of equations $\mathcal{E}$ in one variable such that:

(R) $\vdash_{\mathrm{L}} \varphi \to \varphi$

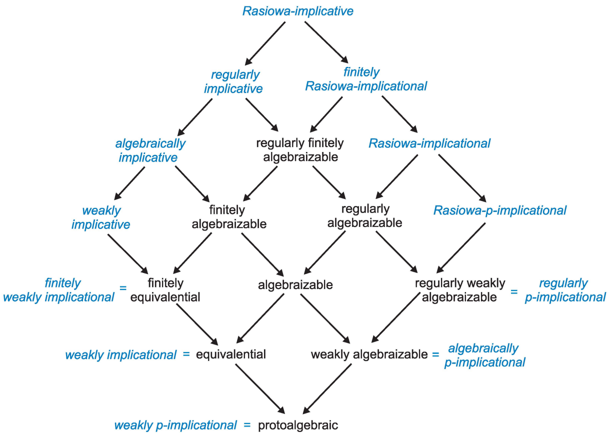
(MP) $\varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi$

(T) $\varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi$

(sCng) $\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} \circ(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n) \to \circ(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n)$

for each $\circ \in \mathcal{O}$ and each $0 \leq i < n$

(Alg) $p \dashv\vdash_{\mathrm{L}} \{\mu(p) \leftrightarrow \nu(p) \mid \mu \approx \nu \in \mathcal{E}\}$.



Rasiowa-implicative
regularly implicative
finitely Rasiowa-implicational
algebraically implicative
regularly finitely algebraizable
Rasiowa-implicational
weakly implicative
finitely algebraizable
regularly algebraizable
Rasiowa-p-implicational
finitely weakly implicational = finitely equivalential
algebraizable
regularly weakly algebraizable = regularly p-implicational
weakly implicational = equivalential
weakly algebraizable = algebraically p-implicational
weakly p-implicational = protoalgebraic

Algebraically implicative logics – 2

Definition 4 ([CN21])

Let L be a logic and $\mathbf{MOD}^*(\mathrm{L})$ is the class of reduced models of L. An algebra $\boldsymbol{A}$ is an L-algebra, $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$ in symbols, if there is a set $F \subseteq A$ such that $\langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$.

Theorem 5 ([CN21])

Let L be a algebraically implicative logic. Then for any set Γ of formulas and any formula φ the following holds:

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi.$$

Algebraically implicative logics – 3

Applying Proposition 2.9.11 in [CN21], we can have the following definition

Definition 6 ([CN21])

A is algebra of truth values for L, or L-algebra, if there is a set of equations $\mathcal{E}$ such that the following quasi-equations hold in **A** for each $\alpha \approx \beta \in \mathcal{E}$:

- $\alpha(\varphi) \approx \beta(\varphi)$, for each axiom φ of L
- $\bigwedge \mathcal{E}[\Gamma] \Rightarrow \alpha(\varphi) \approx \beta(\varphi)$ for each rule $\Gamma \vdash_{\mathrm{L}} \varphi$ of L
- $\bigwedge \mathcal{E}[x \leftrightarrow y] \Rightarrow x \approx y$

Toward first-order language

Definition 7

Let $\mathcal{L}_{\forall\exists} = \langle \mathcal{O}, \forall, \exists, \mathbf{P}, \mathbf{F}, \mathit{Var}, \rho \rangle$ be a first-order language where $\{\rightarrow\} \subseteq \mathcal{O}$ is a set of propositional connectives, $\mathbf{P}(\mathbf{F})$ is a set of relation (functional) symbols, Var is a set of variables, and $\rho : \mathcal{O} \rightarrow \omega$ is an arity function.

First-order algebraically implicative logics – 1

Definition 8 ([CN21])

The minimum first-order logic over an algebraically implicative logic L is the consequence relation $\vdash_{\mathrm{L}\forall^m}$ defined by the following axioms over $\mathcal{L}_{\forall\exists}$-formulas :

(P_{L})	$\tau[\Gamma] \vdash_{\mathrm{L}\forall^m} \tau(\varphi)$	whenever $\Gamma \vdash_{\mathrm{L}} \varphi$
$(lb^{\forall})$	$\vdash_{\mathrm{L}\forall^m} \forall x\varphi(x) \to \varphi(t)$	where t is substitutable for x in φ
$(ub^{\exists})$	$\vdash_{\mathrm{L}\forall^m} \varphi(t) \to \exists x\varphi(x)$	where t is substitutable for x in φ
$(Inf^{\forall})$	$\chi \to \varphi \vdash_{\mathrm{L}\forall^m} \chi \to \forall x\varphi$	where x is not free in χ
$(Sup^{\exists})$	$\varphi \to \chi \vdash_{\mathrm{L}\forall^m} \exists x\varphi \to \chi$	where x is not free in χ
(PGen)	$\varphi(x) \vdash_{\mathrm{L}\forall^m} \varphi(t)$	where t is substitutable for x in φ

First-order algebraically implicative logics – 2

Definition 9 ([CN21])

A structure $\mathfrak{M}$ is a pair $\langle \boldsymbol{A}, \mathbf{M} \rangle$ for a logic L where

- $\boldsymbol{A} \in \mathbf{MOD}^*(\mathrm{L})$
- $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ $(M \neq \emptyset)$
- $P_{\mathbf{M}} \colon M^n \to A$, for each n-ary $P \in \mathbf{P}$ with $n \geq 1$; $P_{\mathbf{M}} \in A$ if P is a propositional constant.
- $f_{\mathbf{M}} \colon M^n \to M$ for each n-ary $f \in \mathbf{F}$ with $n \geq 1$; $f_{\mathbf{M}} \in M$ if f is an object constant.

First-order algebraically implicative logics – 3

Definition 10 ([CN21])

A $\mathfrak{M}$-evaluation v: a mapping $\mathrm{v}\colon Var \to M$. For $x \in Var, m \in M$, and $\mathfrak{M}$-evaluation v, we define $\mathrm{v}[x{\to}m]$ as

$$\mathrm{v}[x{\to}m](x) = m \text{ and } \mathrm{v}[x{\to}m](y) = \mathrm{v}(y) \text{ for } y \neq x.$$

We can inductively extend a $\mathfrak{M}$-evaluation v to all $\mathcal{L}$-terms by setting

$$v^{\mathfrak{M}}(f(t_1, \ldots, t_n)) = f_{\mathbf{M}}(\mathrm{v}^{\mathfrak{M}}(t_1), \ldots, \mathrm{v}^{\mathfrak{M}}(t_n))$$

for each n-ary $f \in \mathbf{F}$.

First-order algebraically implicative logics – 4

Definition 11 ([CN21])

Let L be an algebraically implicative logic. We define values of the terms and truth values of the $\mathcal{L}$-formulas in structure $\mathfrak{M}$ for an evaluation v recursively as

$$\begin{array}{rcll} \|x\|^{\mathfrak{M}}_{\mathrm{v}} & = & \mathrm{v}(x) & \text{for } x \in \mathit{Var} \\ \|P(t_1,\ldots,t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & P_{\mathbf{M}}(\mathrm{v}^{\mathfrak{M}}(t_1),\ldots,\mathrm{v}^{\mathfrak{M}}(t_n)) & \text{for } P \in \mathcal{P} \\ \|\circ(\varphi_1,\ldots,\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}} & = & \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}},\ldots,\|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}) & \text{for } \circ \in \mathcal{O} \\ \|(\forall x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} & = & \inf_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to m]} \mid m \in M\} & \\ \|(\exists x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} & = & \sup_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to m]} \mid m \in M\} & \end{array}$$

If the infimum/supremum does not exist, the value is undefined. A structure $\mathfrak{M}$ is safe if $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}}$ is defined for each φ and v.

Polyadic Ł-Algebras – 1

Definition 12

Let I be a nonempty set. A polyadic $\langle \mathcal{L}_{\forall\exists}, I \rangle$- algebra $\mathbf{A}$ is of the form

$$\langle A, (\circ^{\mathbf{A}} : \circ \in \mathcal{O}), \forall^{\mathbf{A}}, \exists^{\mathbf{A}}, S^{\mathbf{A}} \rangle$$

where $\circ^{\mathbf{A}} : A^n \to A$ if $\rho(\circ) = n$, $\forall^{\mathbf{A}}, \exists^{\mathbf{A}} : \mathcal{P}_\omega(I) \to A^A$, and $S^{\mathbf{A}} : I^I \to A^A$ such that the following axioms are satisfied :

- $S^{\mathbf{A}}_{\iota} x = x$;
- $S^{\mathbf{A}}_{\sigma}(S^{\mathbf{A}}_{\tau} x) = S^{\mathbf{A}}_{\sigma\tau} x$, for all $\sigma, \tau \in I^I$;
- $S^{\mathbf{A}}_{\sigma}(\circ^{\mathbf{A}}(x_1, \ldots, x_{\rho(\circ)})) = \circ^{\mathbf{A}}(S^{\mathbf{A}}_{\sigma} x_1, \ldots, S^{\mathbf{A}}_{\sigma} x_n)$, for all $\circ \in \mathcal{O}$, $\sigma \in I^I$;
- $S^{\mathbf{A}}_{\sigma} Q^{\mathbf{A}}_{J} x = S^{\mathbf{A}}_{\tau} Q^{\mathbf{A}}_{J} x$ for all $Q \in \{\forall, \exists\}$, $J \subseteq_\omega I$, and $\sigma, \tau \in I^I$ such that $\sigma J_* \tau$;
- $Q^{\mathbf{A}}_{J} S^{\mathbf{A}}_{\sigma} x = S^{\mathbf{A}}_{\sigma} Q^{\mathbf{A}}_{\sigma^{-1}(J)} x$ for all $Q \in \{\forall, \exists\}$, $J \subseteq_\omega I$, and $\sigma, \tau \in I^I$ such that σ is injective on $\sigma^{-1}(J)$.

Polyadic L-Algebras – 2

Definition 13

A polyadic $\langle \mathcal{L}_{\forall\exists}, I \rangle$-algebra is called a polyadic L-algebra if it satisfies the following equations and quasi-equations :

- Axioms of L-algebras;
- Axioms (T1)-(T8) for all $\sigma \in I^I$ and $J \subseteq_\omega I$

(T_1) $x \leq y$ implies $S_\sigma x \leq S_\sigma y$;

(T_2) $\forall_\emptyset x = x$;

(T_3) $\exists_\emptyset x = x$;

(T_4) $\forall_J x \leq x$;

(T_5) $x \leq \exists_J x$;

(T_6) $x \leq S_\sigma x$;

(T_7) $x \leq y, \forall_J x = x$ implies $x \leq \forall_J y$;

(T_8) $x \leq y, \exists_J y = y$ implies $\exists_J x \leq y$.

Functional Polyadic L-algebras – 1

Definition 14

A value $\mathcal{L}_{\forall\exists}$-algebra $\mathbf{V}$ is of the form

$$\langle V, (\circ^{\mathbf{V}} : \circ \in \mathcal{O}), \forall^{\mathbf{V}}, \exists^{\mathbf{V}} \rangle$$

where $\circ^{\mathbf{V}} : V^{\rho(\circ)} \to V$ is a $\rho(\circ)$-ary operation on V for each $\circ \in \mathcal{O}$, and $Q^{\mathbf{V}} : \mathcal{P}(V) \rightharpoonup V$ is a partial unary second-order operation on V for each $Q \in \{\forall, \exists\}$.

Functional Polyadic $\mathbb{L}$-algebras – 2

Definition 15

Given a value $\mathcal{L}_{\forall\exists}$-algebra $\mathbf{V}$ and two sets X, I. A *partial functional polyadic* $\langle\mathcal{L}, I\rangle$*- algebra* $\bar{\mathbf{V}}$ is of the form

$$\langle V^{X^I}, (\circ^{\bar{\mathbf{V}}} : \circ \in \mathcal{O}), \forall^{\bar{\mathbf{V}}}, \exists^{\bar{\mathbf{V}}}, S^{\bar{\mathbf{V}}}\rangle$$

where $\circ^{\bar{\mathbf{V}}} : (V^{X^I})^{\rho(\circ)} \to V^{X^I}$, $\forall^{\bar{\mathbf{V}}}, \exists^{\bar{\mathbf{V}}} : \mathcal{P}_\omega(I) \to [V^{X^I}, V^{X^I}]$, and $S^{\bar{\mathbf{V}}} : I^I \to End(\mathbf{V})$ are defined as follows :

- $(\circ^{\bar{\mathbf{V}}}(p_1, \ldots, p_{\rho(\circ)}))(\vec{x}) = \circ^{\mathbf{V}}(p_1(\vec{x}), \ldots, p_{\rho(\circ)}(\vec{x}))$ for all $p_1, \ldots, p_{\rho(\circ)} \in V^{X^I}$ and $\vec{x} \in X^I$;
- $(\forall^{\bar{\mathbf{V}}}_J p)(\vec{x}) = \forall^{\mathbf{V}}(\{p(\vec{y}) : \vec{x} J_* \vec{y}\})$, for all $p \in V^{X^I}$, $J \subseteq_\omega I$, and $\vec{x}, \vec{y} \in X^I$; similarly for $\exists^{\bar{\mathbf{V}}}$
- $(S^{\bar{\mathbf{V}}}_\sigma p)(\vec{x}) = p(\sigma_* x)$ where $(\sigma_* \vec{x})_i = (\vec{x})_{\sigma(i)}$ for all $\sigma \in I^I$ and $\vec{x} \in X^I$.

Functional Polyadic L-algebras – 3

Definition 16

A subalgebra $\bar{\mathbf{U}}$ of $\bar{\mathbf{V}}$ such that $\forall_J^{\bar{\mathbf{V}}} p$ and $\exists_J^{\bar{\mathbf{V}}}$are total functions from X^I to $\mathbf{V}$ is called a functional polyadic $\langle \mathcal{L}_{\forall\exists}, I \rangle$- algebra.

Definition 17

A functional polyadic L-algebra is a total functional polyadic $\langle \mathcal{L}_{\forall\exists}, I \rangle$-algebra whose value algebra is of the form $\langle \mathbf{V}, \forall^{\mathbf{V}}, \exists^{\mathbf{V}} \rangle$ where $\mathbf{V} \in \mathbf{ALG}^*(\mathrm{L})$ and $\forall^{\mathbf{V}}$ and $\exists^{\mathbf{V}}$ are respectively the generalized meet and join operations.

Representation Theorems – 1

Theorem 18

Every functional polyadic L*-algebra is a polyadic* L*-algebra.*

Proof.

Verify all the axioms. □

Representation Theorems – 2

For the converse, as the classical case in [Hal54], we need to make some restrictions. An element a of a polyadic $\langle \mathcal{L}_{\forall\exists}, I \rangle$-algebra has a finite support $J \subseteq I$ if $S_\sigma a = S_\tau a$ for all $\sigma, \tau \in I^I$ such that $\sigma J \tau$. A polyadic $\langle \mathcal{L}_{\forall\exists}, I \rangle$-algebra is locally finite if every element has a finite support.

Representation Theorems – 3

Theorem 19

Every locally finite polyadic L*-algebra of infinite dimension is isomorphic to a functional polyadic* L*-algebra.*

Proof.

Let $\mathbf{A} = \langle A, \rightarrow^{\mathbf{A}} (\circ^{\mathbf{A}} : \circ \in \mathcal{O}), \forall^{\mathbf{A}}, \exists^{\mathbf{A}}, S^{\mathbf{A}} \rangle$ be a locally finite polyadic L-algebra of infinite dimension. By Theorem 1.12 in [PS93], $\mathbf{A}$ is isomorphic to a functional polyadic $\langle \mathcal{L}_{\forall\exists}, I \rangle$-algebra whose domain is I and whose value algebra is $\mathbf{V} = \langle A, \rightarrow^{\mathbf{A}}, (\circ^{\mathbf{A}} : \circ \in \mathcal{O}), \forall^{\mathbf{V}}, \exists^{\mathbf{V}} \rangle$, where $\forall^{\mathbf{V}}, \exists^{\mathbf{V}} : \mathcal{P}(A) \rightharpoonup A$ with the common domain being the set of all subsets of A of the form $\{S_{\lambda}^{\mathbf{A}} a \ : \ \lambda \in I^I, \lambda J_* \sigma\}$ for some $a \in A$, $J \subseteq_\omega I$, and $\sigma \in I^I$. Then show that $\forall^{\mathbf{V}}$ and $\exists^{\mathbf{V}}$ are actually generalized meets and joins. □

More works

Many possible direction for further investigations

- Polyadic relevant algebras and MG structures (jww. Nickolas Ferenz and Andrew Tedder)
- Connection with variable binding algebras[FPT99],[GP02].
- Relation algebras [Ság12]
- Presheaf theoretic formulation (work in progress based on [KP10])
- 2-categorical topoi (based on [LY25] and [Mar21])

Thank you !

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