Properties of the symmetric difference in lattices with complementation

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Outline:

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- Symmetric differences

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- Coincidence identity

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- Coincidence identity
- Associativity of symmetric differences

Symmetric differences

Basic concepts

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Definition 1

(i) *A unary operation on a set L is called an* involution *if it satisfies the identity* $(x')' \approx x$.

(ii) *A unary operation on a poset* $(L, \leq)$ *is called* antitone *if* $x, y \in L$ *and* $x \leq y$ *together imply* $y' \leq x'$.

(iii) *A unary operation on a bounded lattice* $(L, \vee, \wedge, 0, 1)$ *is called a* complementation *if it satisfies the identities* $x \vee x' \approx 1$ *and* $x \wedge x' \approx 0$.

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- *a lattice with complementation if $'$ is a complementation on $(L, \vee, \wedge, 0, 1)$,*
- *an ortholattice if $'$ is a complementation and an antitone involution on $(L, \vee, \wedge, 0, 1)$.*

Symmetric differences

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Lemma 2

A lattice $(L, \vee, \wedge, ', 0, 1)$ *with complementation satisfying the identity* $x \wedge y \approx x \wedge (x' \vee y)$ *or the identity* $x \vee y \approx x \vee (x' \wedge y)$ *satisfies the identity* $(x')' \approx x$.

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$$x +_1 y := (x' \wedge y) \vee (x \wedge y'),$$
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for all $x, y \in L$. *These operations are called symmetric differences.*

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If $(L, \vee, \wedge, ')$ *is a lattice with a unary operation then* $x +_1 y \leq x +_2 y$ *for all* $x, y \in L$.

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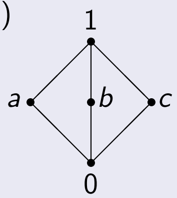
For a lattice $(L, \vee, \wedge, ')$ with a unary operation we call the identity $x +_1 y \approx x +_2 y$ the coincidence identity.

Example

Example

Example 5

The modular lattice $(M_3, \vee, \wedge)$



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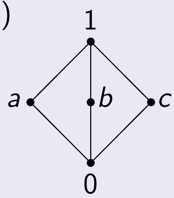

with the complementation $'$ *defined by*

x	0	a	b	c	1
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does not satisfy the coincidence identity since

$$a +_1 b = (a' \wedge b) \vee (a \wedge b') = (b \wedge b) \vee (a \wedge c) = b \vee 0 = b \neq 1 =$$
$$= 1 \wedge 1 = 1 \wedge (b \vee c) = (a \vee b) \wedge (a' \vee b') = a +_2 b.$$

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Example

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Example 8

In Fig. 1 and 2 non-Boolean ortholattices satisfying the coincidence identity are visualized:

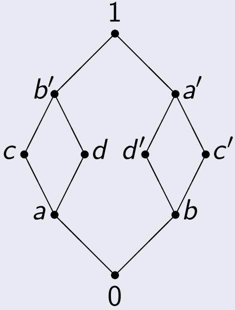

Fig. 1

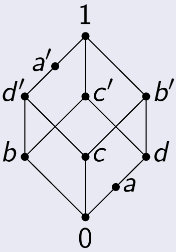

Fig. 2

Example, continued

Example, continued

Example 9

As one can see, in the ortholattice depicted in Fig. 1 the elements c' and d' are incomparable complements of c, but c' and d' are not complements of each other.

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Theorem 10

If $(L, \vee, \wedge, ', 0, 1)$ is a non-trivial lattice with complementation satisfying the coincidence identity and $a \in L$ then there does not exist some $b \in L$ being a complement of a and a'.

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Corollary 11

A lattice with complementation satisfying the coincidence identity cannot contain a subalgebra isomorphic to $(M_3, \vee, \wedge, ', 0, 1)$.

Horizontal sums of bounded chains

Horizontal sums of bounded chains

Theorem 12

Any ortholattice $(L, \vee, \wedge, ', 0, 1)$ *having the property that for all* $x, y \in L$, *either* x *and* y *or* x *and* y' *are comparable with each other satisfies the coincidence identity.*

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Definition 13

If $\mathbf{C}_i = (C_i, \leq_i, 0, 1), i \in I$, *is a family of bounded chains with* $C_i \cap C_j = \{0, 1\}$ *for all* $i, j \in I$ *with* $i \neq j$ *then the bounded lattice*

$$\left(\bigcup_{i\in I} C_i, \bigcup_{i\in I} \leq_i, 0, 1\right)$$

is called the horizontal sum of the $\mathbf{C}_i, i \in I$.

Horizontal sums of two bounded chains

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Theorem 14

If $\mathbf{C}_1 = (C_1, \leq, 0, 1)$ *and* $\mathbf{C}_2 = (C_2, \leq, 0, 1)$ *are bounded chains satisfying* $C_1 \cap C_2 = \{0, 1\}$, $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *denotes the horizontal sum of* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *and* $'$ *is a unary operation on* L *then the following holds:*

(i) *The operation* $'$ *is a complementation if and only if* $0' = 1$, $1' = 0$, $(C_1 \setminus \{0, 1\})' \subseteq C_2 \setminus \{0, 1\}$ *and* $(C_2 \setminus \{0, 1\})' \subseteq C_1 \setminus \{0, 1\}$,

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(ii) *if* $'$ *is a complementation then* $\mathbf{L}$ *satisfies the coincidence identity.*

Example

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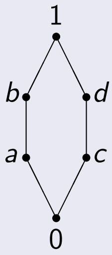
Example 15

The non-modular lattice



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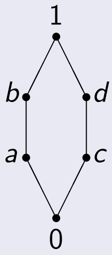

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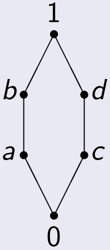

with the complementation ′ defined by

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satisfies the coincidence identity since it is the horizontal sum of two four-element chains.

Example

Example

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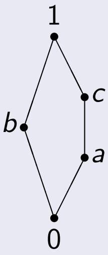

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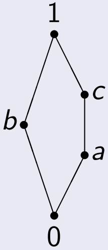

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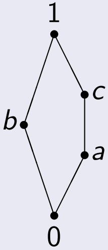

with the complementation $'$ defined by

x	0	a	b	c	1
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satisfies the coincidence identity since it is the horizontal sum of a three-element and a four-element chain.

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For a lattice $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ *with complementation consider the following statements:*

(i) $\mathbf{L}$ *satisfies the identity* $x \wedge y \approx x \wedge (x' \vee y)$*,*

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Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

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Theorem 20

([7]) *For a lattice* $\mathbf{L} = (L, \vee, \wedge, ')$ *with a unary operation the following are equivalent:*

(i) $\mathbf{L}$ *satisfies the identities* $x' \vee (x \wedge y) \approx x' \vee y$ *and* $x' \wedge (x \vee y) \approx x' \wedge y$*,*

(ii) $\mathbf{L}$ *is Boolean.*

Theorem 21

([9], *Proposition 4.2.1*) *For a lattice* $\mathbf{L} = (L, \vee, \wedge, ')$ *with a unary operation the following are equivalent:*

(i) $\mathbf{L}$ *satisfies the identity* $(x \wedge y) \vee (x \wedge y') \approx (x \vee y) \wedge (x \vee y')$*,*

(ii) $\mathbf{L}$ *is Boolean.*

Further characterizations of Boolean lattices

Theorem 20

([7]) *For a lattice* $\mathbf{L} = (L, \vee, \wedge, ')$ *with a unary operation the following are equivalent:*

(i) $\mathbf{L}$ *satisfies the identities* $x' \vee (x \wedge y) \approx x' \vee y$ *and* $x' \wedge (x \vee y) \approx x' \wedge y$,

(ii) $\mathbf{L}$ *is Boolean.*

Theorem 21

([9], *Proposition 4.2.1*) *For a lattice* $\mathbf{L} = (L, \vee, \wedge, ')$ *with a unary operation the following are equivalent:*

(i) $\mathbf{L}$ *satisfies the identity* $(x \wedge y) \vee (x \wedge y') \approx (x \vee y) \wedge (x \vee y')$,

(ii) $\mathbf{L}$ *is Boolean.*

In [6] other single identities are presented that force a lattice with a unary operation to be Boolean.

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Thank you for your attention!