Effect Algebras as a Simplicially Enriched Category

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Two structures

Definition (Foulis and Bennett 1994)

We call a partial algebra $(E, \oplus, ', 0, 1)$ *of signature* $(2, 1, 0, 0)$ *an effect algebra if for each* $a, b, c \in E$*:*

(i) $(E, \oplus, 0)$ *is a partial commutative monoid;*

(ii) a' *is the unique element such that* $a \oplus a' = 1$*,*

(iii) *if* $a \oplus 1$ *is defined, then* $a = 0$ *and* $a \oplus 1 = 1$*.*

An effect algebra admits a natural partial order.
Example: Orthomodular posets, MV-algebras.
Effect algebras with structure-preserving maps form a category **EA**.

Definition

We call a small nonempty category G *a groupoid if each arrow in* G *is an isomorphism.*

Groupoids with functors form a category **GR**.

Both **GR** and **EA** live together in one “hotel”

which we call Frobenius algebras in **Rel**.

Heunen, Contreras, and Cattaneo 2013 (**GR**)
Pavlovic and Seidel 2016 (**EA**).

The rich and the poor

Rich Homo:
Given two groupoids G, H, we can equip the set of natural transformations $\mathrm{Hom}(G, H)$ with a structure of a groupoid.

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Given two effect algebras E, F, the set of morphisms $\mathrm{Hom}(E, F)$ is just a set:
For $f, g \colon E \to F$ (unless in a trivial situation):

- There is no $f \oplus g$ since $f(1) \not\perp g(1)$.
- There is no obvious ordering, since

$$f(a) \leq g(a) \implies f(a') \geq g(a').$$

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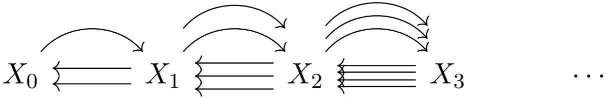
One should somehow enrich the poor EA!

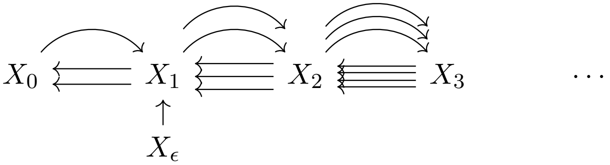
Jump to combinatorial topology

Simplicial sets:



There is a fully faithful functor $N\colon \mathbf{GR} \to \mathrm{Set}_\Delta$.

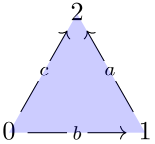
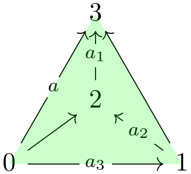
ϵ-simplicial sets



There is a fully faithful functor $N\colon \mathbf{EA} \to \mathrm{Set}_\Delta^\epsilon$. (Lachman 2025)

Effect algebra $E \mapsto \epsilon$-simplicial set $N(E)$

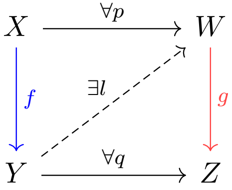
$a = a_1 \oplus \cdots \oplus a_n \quad \longleftrightarrow \quad n$-simplices

$n = 0$	$0 = 0$	the zero element gives the only vertex
$n = 1$	$a = a$	$\{$elements of $E\} \cong \{$edges$\}$ $\{1\} \cong \{\epsilon-$edges$\}$
$n = 2$	$c = a \oplus b$	
$n = 3$	$a = a_1 \oplus a_2 \oplus a_3$	

Lifting properties

For two morphisms $f\colon X \to Y$ and $g\colon W \to Z$ in Set_Δ or $\mathrm{Set}_\Delta^\epsilon$, we set:

$f \boxslash g \iff$



We say g has the *right lifting property (RLP) with respect to* f.

Kan fibration: $g\colon X \to Y$ is called *Kan fibration* if $(\Lambda^n_i \hookrightarrow \Delta^n) \boxslash g$ for each $1 \leq n$, $0 \leq i \leq n$.

Trivial fibration: $g\colon X \to Y$ is called *trivial fibration* if $(\partial\Delta^n \hookrightarrow \Delta^n) \boxslash g$ for each $0 \leq n$.

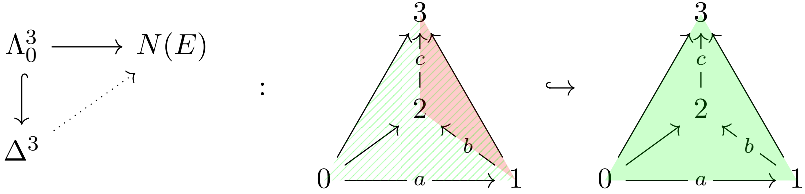
Characterisation theorems

Theorem (folklore)

A simplicial set X is isomorphic to $N(G)$ for a groupoid G iff it admits the unique right lifting property w.r.t. all horns $\Lambda^n_i \subset \Delta^n$, $1 \leq n$, $0 \leq i \leq n$.

Theorem (L.)

*Let $E \in$ **EA**. Then $N(E)$ admits the unique RLP with respect to all horns $\Lambda^n_i \subset \Delta^n$ for $n \geq 3$.*



Theorem (L.)

Let E and F be two effect algebras. Then there is a Frobenius algebra H, such that

$$N(H) \cong \mathrm{Map}(N(E), N(F)). \tag{1}$$

Mapping space

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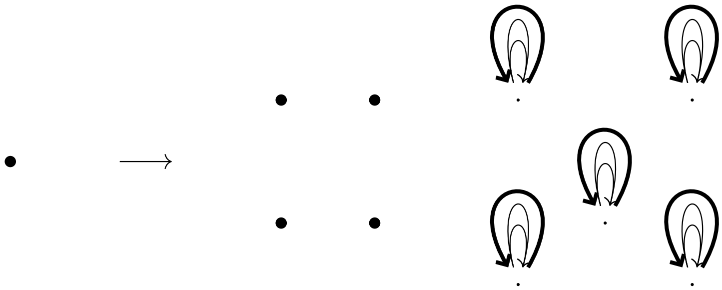
There is a category **EA** *of effect algebras, enriched over the category $\mathrm{Set}_\Delta^\epsilon$ with the cartesian monoidal structure. Moreover, the underlying category* $\mathbf{EA}_0$ *coincides with the ordinary category of effect algebras.*

$$\mathbf{EA}_0(E, F) \cong \mathrm{Hom}(\boldsymbol{\Delta}^0, \mathrm{Map}(N(E), N(F))). \tag{2}$$

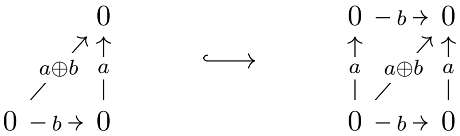
$$\Delta^0 \to \mathrm{Map}(N(E), N(F))$$



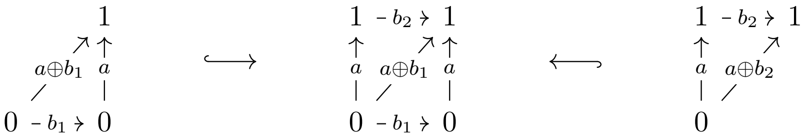
$\{$vertices$\} \cong \{$partial monoid morphisms $E \to F\}$,
$\{$loops over $h\colon E \to F\} \cong$ elements of the interval $[0, h(1)']$.

Pseudo effect algebras (another "hotel" resident)

Commutativity ⇝ *braiding* $\sim$ **EA** ⇝ **PEA**



$$a \oplus b = b \oplus a$$



$$a \oplus b_1 = b_2 \oplus a \tag{3}$$

In this situation, we say that b_1 and b_2 are conjugate.

Let $E, F \in$ **PEA**. In the mapping space $\mathrm{Map}(N(E), N(F))$, we have:

$\{$vertices$\} \cong \{$partial monoid morphisms $E \to F\}$,

arrows between vertices $f\colon E \to F$ and $g\colon E \to F$ corresponds to conjugation.

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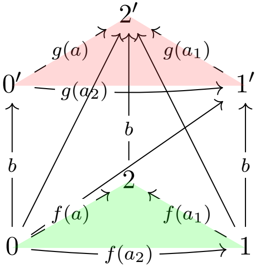
Observation

Let $f\colon E \to F$ be a morphism and g be a pointwise conjugation, that is $b \oplus f(-) = g(-) \oplus b$, for some $b \in F$. Then g is a morphism.

Proof:



$$f(a) = f(a_1) \oplus f(a_2) \implies g(a) = g(a_1) \oplus g(a_2)$$

Main theorem

Theorem (L.)

Let $E, F \in$ **PEA** *and* $f\colon \underline{1} \to E$ *be the unique morphism from the initial (two-elements) PEA* $\underline{1}$*. Then the restriction morphism*

$$p\colon \mathrm{Map}(N(E), N(F)) \to \mathrm{Map}(N(\underline{1}), N(F))$$

is a Kan fibration. Moreover $(\partial\Delta^n \subset \Delta^n) \boxslash p$ *for each* $n \geq 2$*.*

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Thank you for your attention!