Semidirect and poset product of (pseudo-) hoops

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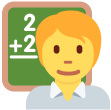
SSAOS 2025
Češkovice
11 September 2025

Motivation



Michal Botur came up with “new product of hoops” and proved that every finite hoop is product of simple hoops.

- How does this relate to “my” semidirect product of residuated structures?
- How does the decomposition relate to the Jipsen–Montagna poset product decomposition of finite hoops?

Pseudo-hoops

- An integral residuated pomonoid is a structure

$$(A, \leq, \cdot, \backslash, /, 1)$$

  where $(A, \leq, \cdot, 1)$ is a pomonoid with top element 1 and, for all $a, b, c \in A$:

$$a \cdot b \leq c \quad \text{iff} \quad a \leq c / b \quad \text{iff} \quad b \leq a \backslash c.$$

- A pseudo-hoop is an integral residuated pomonoid satisfying

$$x \cdot (x \backslash y) \approx (x / y) \cdot y.$$

- The poset of a pseudo-hoop is a meet-semilattice with

$$a \wedge b = a \cdot (a \backslash b) = (a / b) \cdot b.$$

- A hoop is a commutative pseudo-hoop.
- A finite pseudo-hoop is a hoop.

Michal's "new product"

Let C, D be pseudo-hoops and suppose that $\mu\colon C \to D$ satisfies

- $\mu(1) = 1$,
- $\mu(ab) = \mu(a)\mu(b) = \mu(a) \wedge \mu(b)$ for all $a, b \in C$.

Then $C \ltimes_\mu D$ is defined as

$$\{(a, x) \in C \times D \mid \mu(a) \geq x\}$$

with pointwise multiplication and with

$$(a, x) \setminus (b, y) = (a \setminus b, \mu(a \setminus b) \wedge (x \setminus y)),$$
$$(b, y) \,/\, (a, x) = (b \,/\, a, \mu(b \,/\, a) \wedge (y \,/\, x)).$$

Decomposing finite hoops

The product is associative: $(C \ltimes_\mu D) \ltimes_\nu E \cong C \ltimes_M (D \ltimes_N E)$.

Let A be a finite hoop.

- For any filter $D \in \operatorname{Fil} A$,

$$A \cong A/D \ltimes_\mu D$$

  for $\mu\colon A/D \longrightarrow D$ defined by $[a]_D \mapsto a \vee e$, where $D = [e)$.
- Hence A is isomorphic to a product of simple hoops.

“My” semidirect product

Suppose C acts on D by closure endomorphisms, i.e., we are given $\lambda\colon C \curvearrowright D$ such that

- every $\lambda_a\colon x \mapsto a * x$ is a closure endomorphism of D, and
- $1 * x = x$ and $(a \setminus b) * (a * x) = (b \setminus a) * (b * x)$ for all $a, b \in C$ and $x \in D$.

Then we define the semidirect product $C \ltimes_\lambda D$ as

$$\{(a, x) \in C \times D \mid a * x = x\}$$

with

$$(a, x) \setminus (b, y) = (a \setminus b, a * (x \setminus y)),$$
$$(b, y) / (a, x) = (b / a, a * (y / x)),$$

and

$$(a, x) \cdot (b, y) = (ab, ab * xy).$$

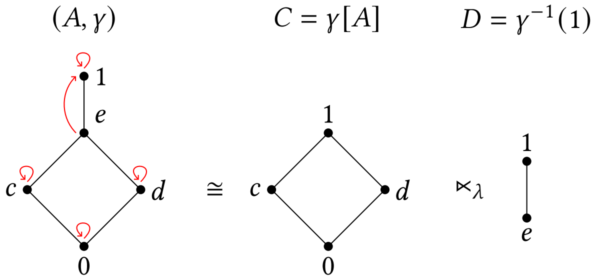
“My” semidirect product

Let A be a pseudo-hoop with a fixed closure endomorphism γ. Then

$$A \cong C \ltimes_\lambda D,$$

where $C = \gamma[A]$ acts on $D = \gamma^{-1}(1)$ by (left) division, i.e., $\lambda \colon C \curvearrowright D$ is given by $\lambda_a(x) = a \setminus x$.

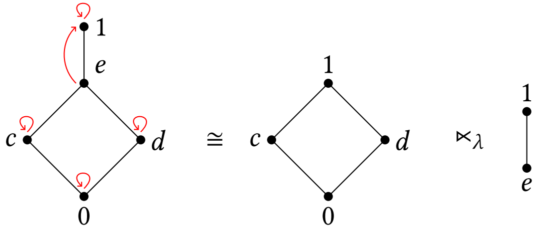
Small example



- $\lambda_a(x) = a \setminus x$: $\lambda_1 = \mathrm{id}_D$ and $\lambda_a = 1_D$ for $a \neq 1$

Small example

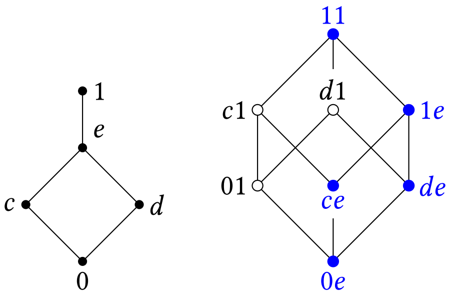
(A, γ) $\quad$ $C = \gamma[A]$ $\quad$ $D = \gamma^{-1}(1)$



- $\lambda_a(x) = a \setminus x$: $\lambda_1 = \mathrm{id}_D$ and $\lambda_a = 1_D$ for $a \neq 1$
- $\mu(a) =$ the bottom of $\lambda_a^{-1}(1)$: $\mu(1) = 1$ and $\mu(a) = e$ for $a \neq 1$

$C \ltimes_\lambda D$ vs $C \ltimes_\mu D$

$$C \ltimes_\lambda D = \{(a,x) \mid a * x = x\} \quad \text{vs} \quad C \ltimes_\mu D = \{(a,x) \mid \mu(a) \geq x\}$$



$$x \mapsto (x'', x \vee e)$$

$C \ltimes_\lambda D$ vs $C \ltimes_\mu D$

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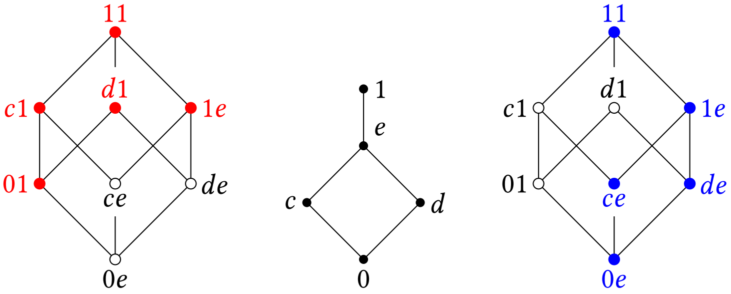


$$(x'', x'' \setminus x) \leftarrow\!\shortmid\ x \mapsto (x'', x \vee e)$$

$C \ltimes_\lambda D$ might seem cumbersome, but it works in setting of the $\{\backslash, /, 1\}$-subreducts (divisible pseudo-BCK-algebras).

$C \ltimes_\mu D \cong C \ltimes_\lambda D$

$C \ltimes_\mu D$ always induces $\lambda\colon C \curvearrowright D$ such that $C \ltimes_\mu D \cong C \ltimes_\lambda D$:

- the elements $\mu(a) \in D$ are idempotent, hence the maps $\lambda_a\colon x \mapsto \mu(a) \setminus x$ are closure endomorphisms of D, with $\lambda_a^{-1}(1) = [\mu(a))$, and
- the map
$$(a, x) \mapsto (a, \mu(a) \setminus x)$$
is an isomorphism $C \ltimes_\mu D \cong C \ltimes_\lambda D$.

$C \ltimes_\lambda D \cong C \ltimes_\mu D$ if C acts on D by "principal" endom.

$C \ltimes_\lambda D$ induces $\mu\colon C \longrightarrow D$ such that $C \ltimes_\lambda D \cong C \ltimes_\mu D$ if the kernels of the closure endomorphisms λ_a are principal filters:

- if $\lambda_a^{-1}(1) = [\mu(a))$ for all $a \in C$, then the map $a \mapsto \mu(a)$ is a "weak morphism", and
- the map
$$(a, x) \mapsto (a, \mu(a) \wedge x)$$
is an isomorphism $C \ltimes_\lambda D \cong C \ltimes_\mu D$.

Poset product [Jipsen 2009, Jipsen–Montagna 2009]

Given $(A_p \mid p \in P)$ a family of bounded pseudo-hoops,
$f \in \prod_P A_p$ is called an antichain labeling if

$$\text{for all } p < q \text{ in } (P, \leq),\ f(p) = 0_p \text{ or } f(q) = 1_q,$$

or, equivalently, if

$$\{p \in P \mid f(p) = 0_p\} \text{ is a down-set, } \{p \in P \mid f(p) = 1_p\} \text{ is an up-set,}$$
$$\text{and } \{p \in P \mid 0_p < f(p) < 1_p\} \text{ is an antichain in } (P, \leq).$$

The poset product $\prod_{(P,\leq)} A_p$ is the set of all antichain labelings with pointwise multiplication and with divisions defined by

$$(f \setminus g)(p) = \begin{cases} f(p) \setminus_p g(p) & \text{if } f(q) \leq_q g(q) \text{ for all } q > p, \\ 0_p & \text{otherwise,} \end{cases}$$

and

$$(g \,/\, f)(p) = \ldots$$

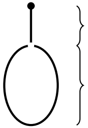
Decomposition of finite hoops [Jipsen–Montagna 2009]

If A is a finite hoop, then

$$A \cong \prod_{(E,\leq^{\mathrm{op}})} B_e \cong \prod_{(P,\subseteq)} U_p,$$

where

- $(E, \leq^{\mathrm{op}})$ is the dual poset of $\vee$-irreducible idempotent elements $e \neq 0$, and for each e, B_e is the interval $[e_*, e]$, where e_* is the lower cover of e;
- $(P, \subseteq)$ is the poset of $\cap$-irreducible filters $p \neq A$, and for each p, U_p is the "upper part" of the ordinal sum decomposition $A/p = L_p \oplus U_p$:



$U_p \cong [e_*, e]$ for $p = [e)$

L_p

$C \ltimes_\mu D$ is poset product (sometimes)

Given $C \ltimes_\mu D$, where

$$C = C_1 \times \cdots \times C_n \quad \text{and} \quad D = \prod_{(S,\leq)} D_s$$

with all the algebras C_i and D_s simple, we extend the partial order from S to $T = \{1, \ldots, n\} \cup S$ by letting, for $i \in \{1, \ldots, n\}$ and $s \in S$,

$$i > s \quad \text{iff} \quad \mu(1_1, \ldots, 0_i, \ldots, 1_n)(s) = 0_s.$$

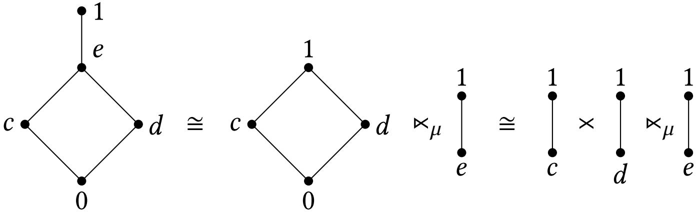
Then

$$C \ltimes_\mu D \cong \prod_{(T,\leq)} A_t,$$

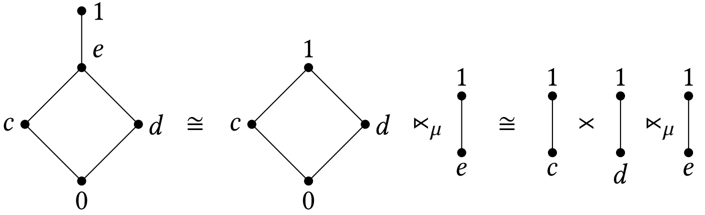
where

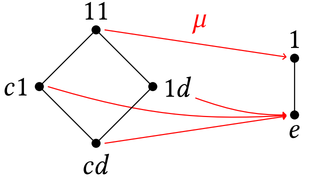
$$A_t = \begin{cases} C_t & \text{for } t \in \{1, \ldots, n\}, \\ D_t & \text{for } t \in S. \end{cases}$$

Example (contd): $C \ltimes_\mu D$ as a poset product

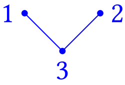


Example (contd): $C \ltimes_\mu D$ as a poset product

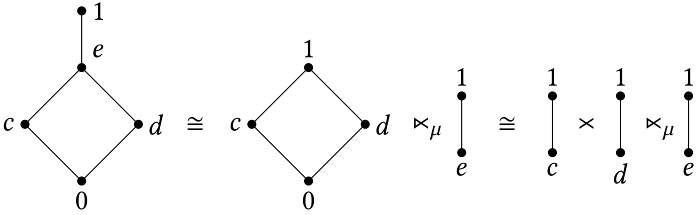


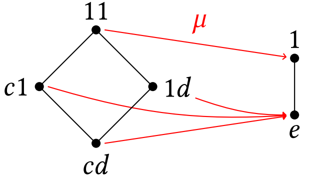


$$i > s \Leftrightarrow \mu(1_1, \ldots, 0_i, \ldots, 1_n)(s) = 0_s$$



Example (contd): $C \ltimes_\mu D$ as a poset product





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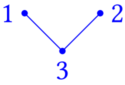

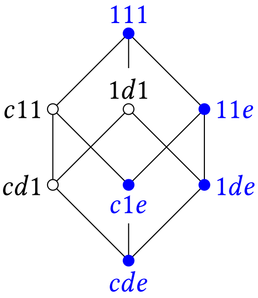

Conclusion

- $C \ltimes_\mu D$ is $C \ltimes_\lambda D$ when C acts on D by "principal" closure endomorphisms
- Decomposition of finite hoops as product of simple hoops is the Jipsen–Montagna poset product decomposition

The end

Thank you!