Orthoset spectra of C*-algebras

David Kruml[1]

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C*-algebras

C*-algebra:

- Banach algebra over $\mathbb{C}$,
- involution,
- $\|a^*a\| = \|a\|^2$,
- assume unit element 1.

Examples:

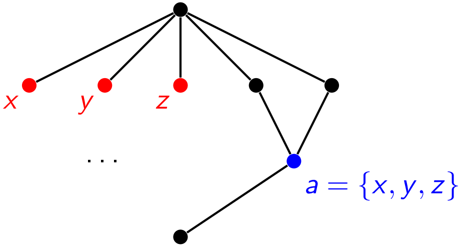
- $\mathbb{C}^n, C(X, \mathbb{C})$ — $\mathbb{C}$-valued continuous functions on compact Hausdorff space,
- $\mathcal{B}(H)$ ... bounded operators on Hilbert space,
- $M_n\mathbb{C}, C(X, M_n\mathbb{C})$ — matrix algebras, matrix-valued cont. functions.

Commutative C*-algebras

Gelfand–Naimark duality:

- Every commutative C*-algebra is of the form $C(X, \mathbb{C})$
- Moreover, there is a duality of categories CC*Alg and KHausTop:
$$\frac{C(X, \mathbb{C}) \to C(Y, \mathbb{C})}{X \leftarrow Y}$$
- The space X can be obtained from frame of closed (in the norm topology) ideals $\mathrm{Idl}\, A$.

Frame of closed ideals



- In Idl A, maximal elements (coatoms) are the only prime elements = points of topology
- $p \in a \;\Leftrightarrow\; a \not\leq p$

q-frames and q-topology

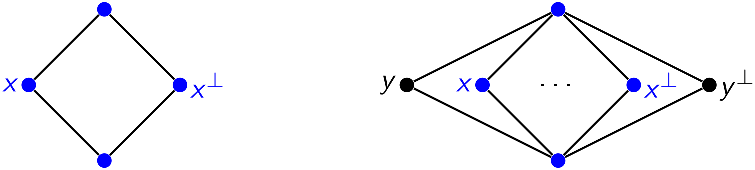
- Closed right-sided ideals $\operatorname{RIdl} A$ (70's, Akemann, Giles, Kummer, Rosický ,...).
- Other attempts: topologies on pure state spaces, quantales of all subspaces (Mulvey).
- $\operatorname{RIdl} A$ (as a lattice) is not a frame and does not determine A.
- However, together with embedding $\operatorname{RIdl} A \to \prod P(H_i)$ it is a full invariant of A.
- Idea of reconstruction:
  - Elements of $\operatorname{RIdl} A$ are assembled to projections on H_i,
  - hermitean element $a \in \mathcal{B}(H)$ is called *q-open* if its carrier is a such projection,
  - and A is determined by its hermitean elements.

q-frames and orthosets

- RIdl A is a "q-frame", it is "spatial" but needs more structure.
- It is very promising to consider orthogonality relation on the points (orthoset, Vetterlein).
- $p \in a \;\Leftrightarrow\; a \not\lesssim p$ and $p \perp q$ for every $q \geq a$
- a is *central* ... $a \not\lesssim p, a \leq q \;\Rightarrow\; p \perp q$.
- Central elements correspond to two-sided ideals.
- *central cover*: $a \mapsto \hat{a}$.
- *quantale structure*: $p \in a, q \in \hat{b}, p \not\perp q \;\Rightarrow\; p \in ab$.
- *quantum frame structure*: $p \in a, q \in b, p \not\perp q \;\Rightarrow\; p, q \in a \circ b$ and $a \circ b$ is central.
- *new operation*: $p \in a, q \in b, p \not\perp q \;\Rightarrow\; p, q \in a \star b$.

Elementary example

$$\mathbb{C}^2 \quad \to \quad M_2\mathbb{C}$$

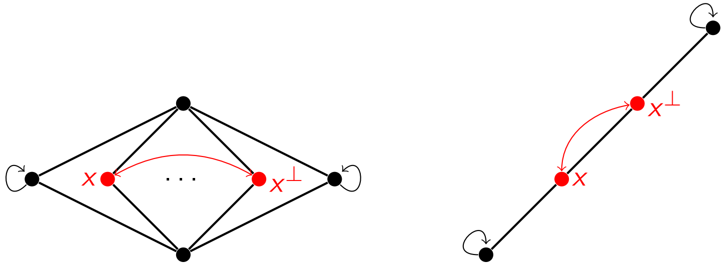


- $\perp$ is preserved and reflected.
- The adjoint "map of q-spaces" must be partial.

Representations

- Every C*-algebra A can be represented in $\mathcal{B}(H)$.
- Irreducible representations are of special interest (correspond to $A \to \mathbb{C}$ in comm. case).
- A has enough irreducible representations $A \to \prod_i \mathcal{B}(H_i)$.
- The irr. reps induce an embedding of $\operatorname{RIdl} A$ to orthomodular lattice $\prod_i L(H_i)$.
- A can be reconstructed from this embedding.
- **The main question:** Could we construct "the correct" embedding just from $\operatorname{RIdl} A$?
- First step is easy: a family of equivalent and orthogonal coatoms = othonormal basis of H_i, and Hilbert space is determined by cardinality of the basis.

Bad automorphisms of RIdl $M_2\mathbb{C}$



- Points p, q, r are *collinear*: $p \wedge q = p \wedge r = q \wedge r$.
- Points p, q are *equivalent* if there is the third point r that p, q, r are collinear.
- The example displays an automorphism of RIdl $M_2\mathbb{C}$ which is not induced by automorphism of $M_2\mathbb{C}$.
- The lines are complex and projective, the picture is only an illustration.

Automorphisms in higher dimensions

- In higher dimensions, there are more lines and they "control" affinity of the automorphisms.
- Collinear automorphisms are always "good".
- The property is known as *fundamental theorem of affine geometry.*

Conclusion

- All irreducible representations except of dimension 2 can be obtained from RIdl $M_2\mathbb{C}$.
- The problem can be reduced to study of *2-homogeneous* C*-algebras (those that have only irr. reps of dim 2).
- The 2-homogeneous C*-algebras are classified by bundle theory.
- Trivial algebras are of the form $C(X, M_2\mathbb{C})$, every algebra is "glued" from a finite number of trivial algebras (thank to compactness of the base space).
- Example of a non-trivial algebra: X is a sphere, consider it is made from two discs, the trivial algebras on the discs are glued by automorphism induced by matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{smallmatrix}\right)$.
- There are no other algebras on the sphere than this and the trivial one.
- Nothing is lost: The trivial algebra has two disjoint non-orthogonal global sections, but the non-trivial does not have, and this difference is still visible on RIdl A.

Thank you for attention!