On fuzzy Galois connections between L-fuzzy posets

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Results:

(1) A pair $(f^{\rightarrow}, f^{\leftarrow})$ of maps is a fuzzy Galois connection between (L^X, S_X) and (L^Y, S_Y): $f^{\rightarrow} \dashv f^{\leftarrow}$, i.e.,

$$S_Y(f^{\rightarrow}A, B) = S_X(A, f^{\leftarrow}B) \quad (\forall A \in L^X, B \in L^Y).$$

(2) A pair $(\sqcup, \downarrow)$ of maps is a fuzzy Galois connection between (L^X, S) and (X, e): $\sqcup \dashv \downarrow$, i.e.,

$$e(\sqcup A, t) = S(A, \downarrow t) \quad (\forall t \in X,\ A \in L^X),$$

(3) For a map $f : (X, e_X) \to (Y, e_Y)$, we have

$$\exists g : (Y, e_Y) \to (X, e_X); f \dashv g$$
$$\iff \forall y \in Y\ \exists\, a \in X\, ; f^{\leftarrow}(\downarrow y) = \downarrow a.$$

(complete) residuated lattice

Let $\mathcal{L} =< L, \wedge, \vee, \odot, 0, 1 >$ be a complete residuated lattice, i.e.,

(i) $< L, \wedge, \vee, 0, 1 >$ is a complete lattice;

(ii) $< L, \odot, 1 >$ is a commutative monoid;

(iii) For all $a, b, c \in L$,

$$a \odot b \leq c \iff a \leq b \to c.$$

Proposition 1.([3]) For all $a, b, c, a_i, b_i \in L$, we have

(1) $a \odot a' = 0$, where $a' = a \to 0$;

(2) $a \leq b \iff a \to b = 1$;

(3) $a \odot (a \to b) \leq b$;

(4) $a \leq b \implies a \odot c \leq b \odot c, c \to a \leq c \to b,$
$b \to c \leq a \to c$;

(5) $1 \to a = a$;

(6) $a \vee (b \to c) \leq b \to a \vee c$;

(7) $a \odot (\bigvee_i b_i) = \bigvee_i (a \odot b_i)$;

(8) $(\bigvee_i a_i)' = \bigwedge_i a_i'$;

(9) $a \to (\bigwedge_i b_i) = \bigwedge_i (a \to b_i)$, $(\bigvee_i a_i) \to b = \bigwedge_i (a_i \to b)$.

L-fuzzy poset (X, e)

(X, e) is called an L-fuzzy poset if X is a non-empty set and $e : X \times X \to L$ (L-fuzzy partial order) is a map satisfying:

(e1) $e(x, x) = 1$;

(e2) $e(x, y) = e(y, x) = 1 \;\Rightarrow\; x = y$;

(e3) $e(x, y) \odot e(y, z) \leq e(x, z)$.

$$\Downarrow$$

$$e(x, t) = \bigwedge_{s \in X} (e(s, x) \to e(s, t))$$

Example For a complete residuated lattice L and a non-empty set X, we define

$$L^X = \{A \mid A : X \to L\}.$$

For L^X, we define a map $S : L^X \times L^X \to L$ by

$$S(A, B) = \bigwedge_{x \in X} (A(x) \to B(x)) \quad (A, B \in L^X).$$

$$\Downarrow$$

From the definition, we have

(e1) $S(A, A) = 1$;

(e2) $S(A, B) = S(B, A) = 1 \;\Rightarrow\; A = B$;

(e3) $S(A, B) \odot S(B, C) \leq S(A, C)$.

$$\Downarrow$$

(L^X, S) is an L-fuzzy poset.

Let (X, e_X) and (Y, e_Y) be L-fuzzy posets.

A map $f : (X, e_X) \to (Y, e_Y)$ is called **order-preserving** if

$$e_X(x_1, x_2) \leq e_Y(f(x_1), f(x_2)) \quad (\forall\, x_1, x_2 \in X).$$

For $f : (X, e_X) \to (Y, e_Y)$ and $g : (Y, e_Y) \to (X, e_X)$, a pair (f, g) is called a **fuzzy Galois connection (fGc)** $(f \dashv g)$ if

$$e_Y(fx, y) = e_X(x, gy) \quad (\forall x \in X,\ \forall y \in Y).$$

Let (X, e) be an L-fuzzy poset and $a \in X$.

We define special maps $\uparrow a, \downarrow a : X \to L \quad$ (i.e., $\uparrow a, \downarrow a \in L^X$) by

$$(\uparrow a)(x) = e(a, x)\,;$$

$$(\downarrow a)(x) = e(x, a)\,.$$

For a map $f : (X, e_X) \to (Y, e_Y)$, we also define

$f^{\to} : (L^X, S_X) \to (L^Y, S_Y)$ and $f^{\leftarrow} : (L^Y, S_Y) \to (L^X, S_X)$ by

$$(f^{\to}A)(y) = \bigvee_{f(x)=y} A(x) \ \ (y \in Y,\ A \in L^X);$$

$$(f^{\leftarrow}B)(x) = B(f(x)) \ \ (x \in X,\ B \in L^Y).$$

$$\Downarrow$$

$$\begin{aligned} f \dashv g &\iff e_X(x, gy) = e_Y(fx, y) \ \ (\forall x, y \in X) \\ &\iff \downarrow(gy)(x) = (\downarrow y)(fx) = f^{\leftarrow}(\downarrow y)(x) \ \ (\forall x, y \in X) \\ &\iff \downarrow(gy) = f^{\leftarrow}(\downarrow y) \ \ (\forall y \in X) \\ &\iff \uparrow(fx) = g^{\leftarrow}(\uparrow x) \ \ (\forall x \in X). \end{aligned}$$

Lemma 2. For a map $f:(X,e_X)\to(Y,e_Y)$, we have

(i) $f^{\to}$ and $f^{\leftarrow}$ are order-preserving;

(ii) $S_X(A, f^{\leftarrow}(f^{\to}A)) = 1$ i.e., $Id_{L^X} \leq f^{\leftarrow}f^{\to}$;

(iii) $B = f^{\to}(f^{\leftarrow}B) \quad (\forall\, B \in L^Y)$, thus $f^{\to}f^{\leftarrow} = Id_{L^Y}$.

$$\Downarrow$$

Theorem 3.

A pair $(f^{\rightarrow}, f^{\leftarrow})$ of maps is a fuzzy Galois connection between (L^X, S_X) and (L^Y, S_Y), i.e.,

$$S_Y(f^{\rightarrow}A, B) = S_X(A, f^{\leftarrow}B) \quad (\forall A \in L^X, B \in L^Y).$$

$$f^{\rightarrow} \dashv f^{\leftarrow}$$

Let (X, e) be an L-fuzzy poset and $A \in L^X$.

An element $x_0 \in X$ is called a **join** of A denoted by

$$x_0 = \sqcup A$$

if

(1) $A(x) \leq e(x, x_0) \quad (\forall x \in X)$, hence $e(x, \sqcup \downarrow x) = 1$;

(2) $\bigwedge_{x \in X}(A(x) \to e(x, t)) \leq e(x_0, t) \quad (\forall t \in X)$.

Proposition ([5]) If $\sqcup A$ exists, then we have

$$e(\sqcup A, t) = \bigwedge_{x \in X}(A(x) \to e(x, t)) \ \ (\forall\, t \in X).$$

In the following, we assume that $\sqcup A$ exists for any $A \in L^X$.

We have a new fuzzy Galois connection between (L^X, S) and (X, e).

Theorem 4. Let (X, e) be an L-fuzzy poset. For the map $\sqcup : (L^X, S) \to (X, e)$, we have

$$S(A_1, A_2) \leq e(\sqcup A_1, \sqcup A_2) \ \ (\forall A_1, A_2 \in L^X),$$

Moreover,

$$e(\sqcup A, t) = S(A, \downarrow t) \ (\forall t \in X, \ A \in L^X),$$

i.e., $\sqcup \dashv \downarrow$.

Corollary

(1) $S(A, \downarrow \sqcup A) = 1 \quad (\forall A \in L^X)$;

(2) $e(x, \sqcup \downarrow x) = e(\sqcup(\downarrow x), x) = 1 \quad (\forall x \in X)$, hence

$$\sqcup \downarrow = Id_X.$$

Corollary For all $x, t \in X$,

$$e(x, t) = S(\downarrow x, \downarrow t) = S(\uparrow t, \uparrow x).$$

Theorem (Yao and Lu, 2009) Let $f : (X, e_X) \to (Y, e_Y)$ be a map.

f is order-preserving and has a right adjoint
(i.e., $f \dashv g$ for some g)

$\iff \quad f(\sqcup A) = \sqcup f^{\to}(A) \quad (\forall A \in L^X)$.

We give a simple characterization of fuzzy Galois connections, which says that the assumption of order-preserving of f above is redundant.

We recall the definition $f \dashv g$ for $f : (X, e_X) \to (Y, e_Y)$ and $g : (Y, e_Y) \to (X, e_X)$.

It has several representations as follows:

$$
\begin{aligned}
f \dashv g \iff & e_X(x, gy) = e_Y(fx, y) \\
\iff & \downarrow(gy)(x) = (\downarrow y)(fx) = f^{\leftarrow}(\downarrow y)(x) \\
\iff & \downarrow(gy) = f^{\leftarrow}(\downarrow y) \\
\iff & \uparrow(fx) = g^{\leftarrow}(\uparrow x).
\end{aligned}
$$

$$\Downarrow$$

Theorem 5. Let L be a residuated lattice and $f : (X, e_X) \to (Y, e_Y)$ a map between (X, e_X) and (Y, e_Y). Then

$$\exists g : (Y, e_Y) \to (X, e_X) \ \text{ s.t. } f \dashv g$$

$$\iff \ f(\sqcup A) = \sqcup f^{\to}(A) \quad (\forall A \in L^X).$$

$$\begin{array}{ccc}
(X, e_x) & \xrightarrow{\ f\ } & (Y, e_Y) \\
\sqcup \uparrow & \circlearrowright & \uparrow \sqcup \\
(L^X, S_x) & \xrightarrow[f^{\to}]{} & (L^Y, S_Y)
\end{array}$$

Proof. $(\Leftarrow)$ We assume $f(\sqcup A) = \sqcup f^{\rightarrow}(A)$ for all $A \in L^X$. We define a map $g : (Y, e_Y) \to (X, e_X)$ by

$$g(y) = \sqcup f^{\leftarrow}(\downarrow y) \quad (\forall\, y \in Y).$$

Since $S_X(A, \downarrow \sqcup A) = e_X(\sqcup A, \sqcup A) = 1$ for any $A \in L^X$, regarding A as $f^{\leftarrow}(\downarrow y)$, we get

$$S_X(f^{\leftarrow}(\downarrow y), \downarrow \sqcup(f^{\leftarrow}(\downarrow y))) = 1.$$

On the other hand,

$$
\begin{aligned}
S_X(\downarrow \sqcup(f^{\leftarrow}(\downarrow y)), f^{\leftarrow}(\downarrow y)) &\overset{f^{\rightarrow}\dashv f^{\leftarrow}}{=} S_Y(f^{\rightarrow}(\downarrow \sqcup(f^{\leftarrow}(\downarrow y))), \downarrow y) \\
&\overset{\sqcup\dashv\downarrow}{=} e_Y(\sqcup f^{\rightarrow}(\downarrow \sqcup(f^{\leftarrow}(\downarrow y))), y) \\
&\overset{assumption}{=} e_Y(f(\sqcup(\downarrow \sqcup(f^{\leftarrow}(\downarrow y)))), y) \\
&\overset{\sqcup\downarrow=I}{=} e_Y(f(\sqcup(f^{\leftarrow}(\downarrow y))), y) \\
&\overset{assumption}{=} e_Y(\sqcup(f^{\rightarrow}f^{\leftarrow}(\downarrow y)), y) \\
&\overset{f^{\rightarrow}f^{\leftarrow}=Id}{=} e_Y(\sqcup \downarrow y, y) \overset{\sqcup\downarrow=Id}{=} e_Y(y, y) = 1.
\end{aligned}
$$

Therefore, we have

$$
f^{\leftarrow}(\downarrow y) = \downarrow \sqcup(f^{\leftarrow}(\downarrow y)) = \downarrow (gy) \qquad \therefore \; f \dashv g.
$$

($\Rightarrow$) We suppose that $f \dashv g$ for some $g : (Y, e_Y) \to (X, e_X)$. Since

$$
\begin{aligned}
e_Y(f(\sqcup A), y) &\overset{f \dashv g}{=} e_X(\sqcup A, gy) \\
&\overset{\sqcup \dashv \downarrow}{=} S_X(A, \downarrow(gy)) = S_X(A, f^{\leftarrow}(\downarrow y)) \\
&\overset{f^{\rightarrow} \dashv f^{\leftarrow}}{=} S_Y(f^{\rightarrow}A, \downarrow y) \\
&\overset{\sqcup \dashv \downarrow}{=} e_Y(\sqcup f^{\rightarrow}A, y)
\end{aligned}
$$

$$\therefore \ e_Y(f(\sqcup A), y) = e_Y(\sqcup f^{\rightarrow}A, y) \quad (\forall y \in Y)$$

This implies

$$f(\sqcup A) = \sqcup f^{\rightarrow}A \quad (\forall A \in L^X).$$

We give another simple condition for the existence of the right adjoint.

Theorem 6. Let $f : (X, e_X) \to (Y, e_Y)$ be a map. Then

$$\exists g : (Y, e_Y) \to (X, e_X) \text{ s.t. } f \dashv g$$
$$\iff \forall y \in Y \ \exists a \in X \text{ s.t. } f^{\leftarrow}(\downarrow y) = \downarrow a.$$

Roughly, every inverse image of down-set $\downarrow y$ can be represented by a certain down-set $\downarrow a$ of X.

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Thank you for your attention!