About the lattice of sub(quasi)varieties of the class of pointed Abelian ℓ-groups

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Abelian ℓ-groups

Definition

We say that $\mathbf{A} = \langle A, +, -, \vee, \wedge, 0\rangle$ *is an Abelian* ℓ*-group if* $\langle A, +, -, 0\rangle$ *is an Abelian group,* $\langle A, \vee, \wedge\rangle$ *is a lattice and* $\mathbf{A}$ *satisfies monotonicity condition, that means* $x \leq y$ *implies* $x + z \leq y + z$*. We denote the class of Abelian* ℓ*-groups by* $\mathbb{AL}$*.*

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Corollary

$\mathbb{AL}$ *does not contain any notrivial subquasivarieties.*

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Definition (Lexicographic product)

For totally ordered ℓ*-groups* $\mathbf{A}, \mathbf{B}$ *we define lexicographic product* $\mathbf{A} \overrightarrow{\times} \mathbf{B}$ *as a product* $\mathbf{A} \times \mathbf{B}$ *with the redefined ordering as follows:*
$\langle a_1, b_1 \rangle \leq \langle a_2, b_2 \rangle \iff (a_1 < a_2) \vee (a_1 = a_2 \wedge b_1 \leq b_2)$.

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$\Gamma : \{\mathbf{A}_a \mid \mathbf{A} \in \mathbb{AL}, (\forall b \in A)(\exists n \in \mathbb{N})\ n \cdot a \geq b\} \to \mathsf{MV}$
$\Gamma : \mathbf{A}_a \mapsto \mathbf{A}_a \upharpoonright [0, a]$.

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The functors Γ and Γ^{-1} is preserving $\mathbf{H}, \mathbf{S}, \mathbf{P}$ and partial embeddings.

Strongly pointed Abelian ℓ-groups

Lemma

$p\mathbb{AL}^0 = \{\mathbf{A}_0 \mid \mathbf{A} \in \mathbb{AL}\}$ is the smallest nontrivial subvariety of $p\mathbb{AL}$. Alternatively, we can say that any non-trivial proper subvariety of $p\mathbb{AL}$ contains $p\mathbb{AL}^0$ as a subvariety.

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Lemma

Let $\mathbf{A}_b$ be a finitely generated totally ordered pointed Abelian ℓ-group and $\mathbf{B}_b$ be its convex pointed ℓ-subgroup with strong unit $0 \neq b \in B$. Then $\mathrm{ISP_U}(\mathbf{A}_b) = \mathrm{ISP_U}(\mathbf{B}_b)$.

Komori classification

Theorem

Every proper subvariety of MV-algebras is equal to $\mathbf{HSP}(\{Ł_i\}_{i \in I} \cup \{K_j\}_{j \in J})$ *for some finite sets* $I, J \subseteq \mathbb{N} \setminus \{0\}$.

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Theorem

Every proper relative subvariety of positively pointed Abelian ℓ*-groups is generated by* $\{\boldsymbol{Z}_i\}_{i \in I} \cup \{\boldsymbol{Z}_j \overrightarrow{\times} \boldsymbol{Z}_0\}_{j \in J}$ *for some finite sets* $I, J \subseteq \mathbb{N}$.

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This approach is not giving any axiomatization!

The equations used for axiomatization

$$(\mathtt{f} - n \cdot x) \vee -x \geq 0 \tag{s-rank$_n$}$$

$$((2n+1) \cdot x - 2 \cdot f) \vee (f - (2n+2) \cdot x) \vee -x \geq 0. \tag{rank$_n$}$$

$$((k+1) \cdot ((p \cdot x - \mathtt{f}) \vee (\mathtt{f} - p \cdot x)) - \mathtt{f}) \vee -x \geq 0 \tag{div$_{p,k}$}$$

$$(n \cdot ((p \cdot x - \mathtt{f}) \vee (-x) \vee (\mathtt{f} - p \cdot x)) - \mathtt{f}) \vee (n \cdot y - \mathtt{f}) \vee (-y) \geq 0 \tag{mix$_{p,n}$}$$

Axiomatization of axiomatic extensions of subvarieties of positively pointed Abelian ℓ-groups

Theorem

Any proper subvariety of $\mathrm{p}\mathbb{AL}^{+}$ *is of the form*

$$V_{I,J} = \mathrm{HSP}(\boldsymbol{Z}_i, \boldsymbol{Z}_j \overrightarrow{\times} \boldsymbol{Z}_0 \mid i \in I, j \in J)$$

for some finite sets $J \subseteq I \subsetneq \mathbb{N}$.
Moreover, $V_{I,J}$ *is generated by the following set* $S_{I,J}$ *of equations:*

$$S_{I,J} = \{(\mathrm{rank_n})\} \cup \{(\mathrm{div_{p,n}}) \mid p \notin I\} \cup \{(\mathrm{mix_{p,n}}) \mid p \in I \setminus J\},$$

where $n = \max I$.

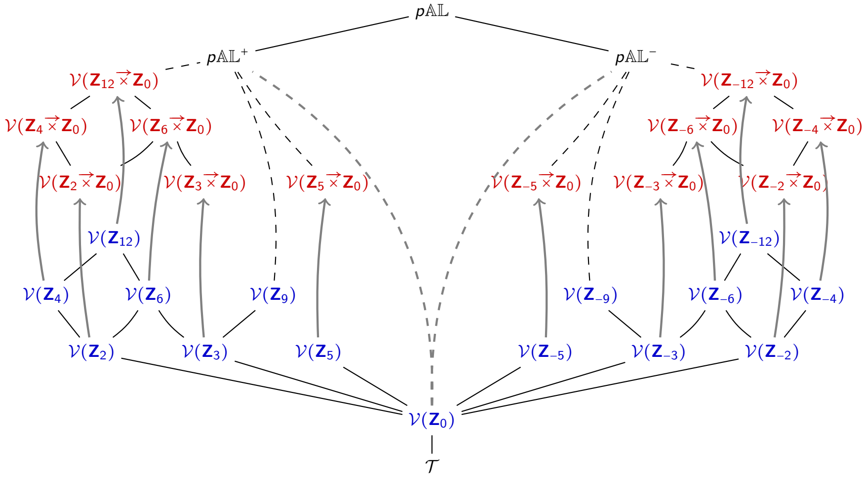

Figure: A part of the lattice of join irreducible subvarieties of pAb.

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Theorem (Gispert 2002)

Let $\mathbf{S}$ *denote any finitely generated dense* ℓ*-subgroup of* $\mathbf{R}$ *such that* $\mathbf{S} \cap \mathbf{Q} = \mathbf{Z}$*. Every subquasivariety of MV-algebras generated by chains is equal to*

$$\mathrm{ISPP_U}(\{\mathbf{L}_n \mid n \in A\} \cup \{\Gamma(\boldsymbol{Z}_n \overrightarrow{\times} \boldsymbol{Z}_m) \mid n \in B, m \in \gamma(n) \cup \{\Gamma(\mathbf{S}_d) \mid d \in C\}\}),$$

for some $A, B, C \subseteq \mathbb{N} \setminus \{0\}$*, and* $\gamma : n \mapsto \gamma(n) \subseteq \mathrm{div}(n)$*.*

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