Meet-irreducibility of congruence lattices of prime-cycled algebras

Juraj Hirjak, Lucia Kőszegyová

Pavol Jozef Šafárik University in Košice

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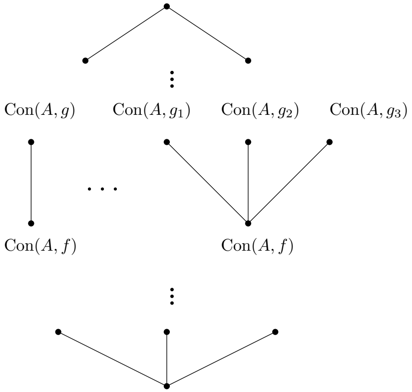
Content

Introduction

- The set of all congruences on a algebra (A, F) (ordered by inclusion) forms a lattice, denoted by $\mathbf{Con(A, F)}$.
- The set of all congruence lattices of all algebras defined on a fixed base set A forms a lattice, denoted $\boldsymbol{\mathcal{E}_A}$. I.e. $\mathcal{E}_A = \{\mathrm{Con}(A, F) : F \subseteq A^A\}$
- Let L be a lattice. A nonunit element $a \in L$ is called **meet-irreducible** (shortly $\wedge$-irreducible) if $a = b_1 \wedge b_2$ implies $a \in \{b_1, b_2\}$.

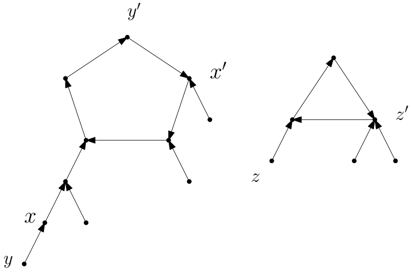
Introduction

We denote monounary algebra: (A, f)



Introduction

- Distance from cycle: $t_f(a)$
- For $x, y \in A$ let $\theta_f(x, y)$ denote the smallest congruence of (A, f) such that $(x, y) \in \theta_f(x, y)$.
- A cyclic element x' is a colleague of x iff $f^{t_f(x')} = f^{t_f(x)}$.



Preliminaries

Lemma 1

Let $f, g \in A^A$ *be nontrivial operations such that* $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$*. Then we have*

1. $\forall x, y \in A : (x, y) \in \alpha \in \mathrm{Con}(A, f) \Longrightarrow (g(x), g(y)) \in \alpha$, *in particular we have* $(g(x), g(y)) \in \theta_f(x, y)$ *and* $\theta_g(x, y) \subseteq \theta_f(x, y)$.
2. *Let* B *be a subalgebra of* (A, f)*. Then either* B *is also a subalgebra of* (A, g) *or* g *is constant on* B*, where the constant does not belong to* B*.*

Corollary 2

Let $g_i, i \in I$*, be nontrivial operations on* A*. Then*

$$\mathrm{Con}(A, f) = \bigcap_{i \in I} \mathrm{Con}(A, g_i) \Longleftrightarrow \forall x, y \in A : \theta_f(x, y) = \bigvee_{i \in I} \theta_{g_i}(x, y).$$

Preliminaries

$$\mathrm{Con}(A,f)=\bigcap_{i\in I}\mathrm{Con}(A,g_i)\iff\forall x,y\in A:\theta_f(x,y)=\bigvee_{i\in I}\theta_{g_i}(x,y)$$

How to prove meet-reducibility:
find $g_i, i \in I$, verify that the equation holds

How to prove meet-irreducibility:
prove that there are no such $g_i, i \in I$

Preliminaries

Known:

connected algebras, algebras with small cycles, algebras with short tails

Unknown:

non-connected algebras with at least one cycle with at least 3 elements and there exists element x such that $f(x)$ is nocyclic

Aim

In this talk, we will focus on prime-cycled algebras.

Definition 3

Let (A, f) be a monounary algebra. (A, f) is said to be a **prime-cycled algebra** if each cycle of (A, f) contains a prime number of elements.

Prime-cycled algebras

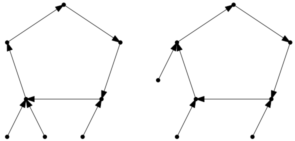
Theorem 4

Let (A, f) be a prime-cycled algebra such that each cycle of (A, f) is a p-cycle. $\mathrm{Con}(A, f)$ is $\wedge$-irreducible in $\mathcal{E}_A$ iff one of the following holds:

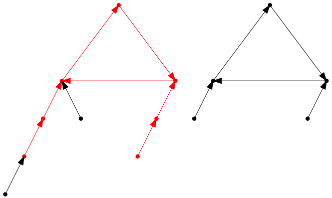
1. *(A, f) is a permutation algebra with $|A| = 2$, or*
2. *(A, f) is a permutation algebra with short tails such that $|A| \geq 3$ and there are at least two cycles in (A, f), or*
3. *(A, f) contains a connected subalgebra B such that there is $x \in B$ with $t_f(x) \geq 2$ and $\mathrm{Con}(B, f \restriction B)$ is $\wedge$-irreducible in $\mathcal{E}_B$, or*
4. *(A, f) is non-connected algebra and there are distinct noncyclic elements $a, b, c, d \in A$ such that $f(a), f(c)$ are cyclic, $f(b) = a$, $f(d) = c$ and $f(a) \neq f(c)$, or*
5. *(A, f) is non-connected algebra and there are distinct noncyclic elements $a, b, c, d, e \in A$ such that $f(a), f(c)$ are cyclic, $f(b) = a$, $f(d) = c$, $f(e) = d$ and $f(a) = f(c)$.*

Prime-cycled algebras

Examples of algebras

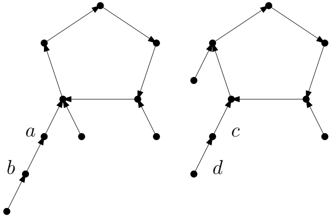


(a) meet-irreducible

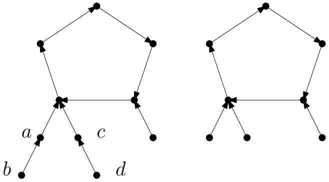


(b) meet-irreducible

Prime-cycled algebras



(a) meet-irreducible

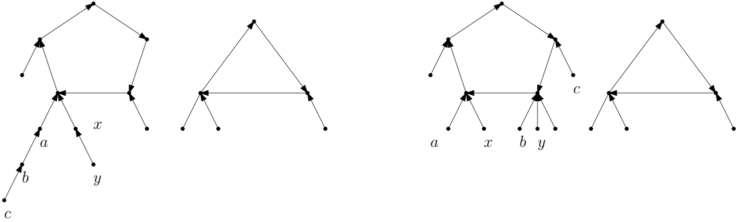


(b) meet-reducible

New results

Lemma 5

Let (A, f) *be a monounary algebra and* $g \in A^A: \ g(x) = f(x')$*.*
Then for every $x, y \in A: \ \theta_g(x,y) \subseteq \theta_f(x,y)$*.*

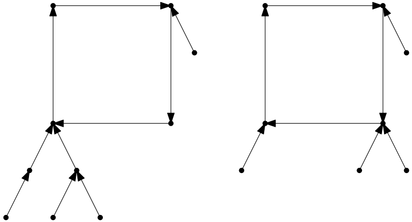


(a) $f(x)$ (b) $g(x)$

New results

Proposition 6

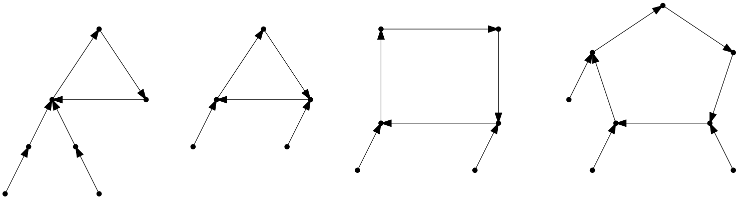
Let (A, f) be a algebra with at least one long tail, such that each cycle of (A, f) is a n-cycle, $n \geq 2, n \in \mathbb{N}$ and $t_f(x) \leq 2$ for every $x \in A$. If for every $x, y \in A : t_f(x) = t_f(y) = 2$ implies that $f^2(x) = f^2(y)$, then $\operatorname{Con}(A, f)$ is $\wedge$-reducible in $\mathcal{E}_A$.



New results

Proposition 7

Let (A, f) be a monounary algebra such that it contains subalgebras B and $C = A \setminus B$. Let $(B, f \restriction B)$ be a algebra with at least one long tail, such that each cycle of $(B, f \restriction B)$ is a n-cycle, $n \geq 2$ and $t_f(x) \leq 2$ for every $x \in B$. Let $(C, f \restriction C)$ be algebra with short tails. If for every $x, y \in A : t_f(x) = t_f(y) = 2$ implies that $f^2(x) = f^2(y)$, then $\mathrm{Con}(A, f)$ is $\wedge$-reducible in $\mathcal{E}_A$.

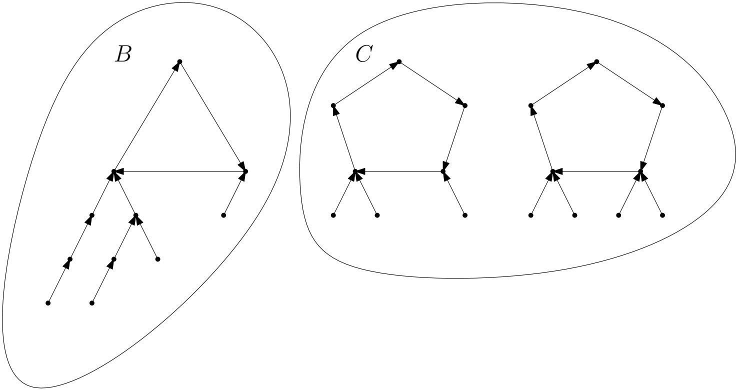


New results

Proposition 8

Let (A, f) be a monounary algebra such that it contains subalgebras B and $C = A \setminus B$. Let $(B, f \restriction B)$ be a conected algebra with at least one long tail, such that its cycle is a p-cycle, p is odd prime and $t_f(x) \leq p$ for every $x \in B$. Let $(C, f \restriction C)$ be algebra with short tails, such that each cycle of $(C, f \restriction C)$ has prime length and these lengths are coprime with p. If for every $x, y \in A : t_f(x) = t_f(y) = 2$ implies that $f^2(x) = f^2(y)$, then $\mathrm{Con}(A, f)$ is $\wedge$-reducible in $\mathcal{E}_A$.

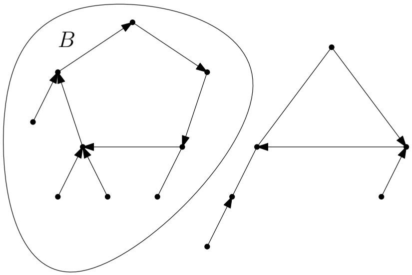
New results



New results

Proposition 9

Let (A, f) be a prime-cycled algebra with a component B such that $(B, f \restriction B)$ is a permutation-algebra with short tails and $p \geq 3$ cyclic elements. If there is no other cycle of (A, f) with p elements, then $\mathrm{Con}(A, f)$ is $\wedge$-reducible in $\mathcal{E}_A$.



Sources

1. Jakubíková-Studenovská D., Pöschel R., Radeleczki S., 2018. The lattice of congruence lattices of algebras on a finite set. In: *Algebra Universalis*. Vol. 79(4). ISSN 1420-8911.
2. Jakubíková-Studenovská D., Janičková L., 2018. Meet-irreducible congruence lattices. In: *Algebra Universalis*. Vol. 79(89). ISSN 1420-8911.
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Thank you for your attention.