On algebras with easy direct limits

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Outline

1. Introduction
   - Retracts
   - Direct limits
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3. A mono-unary EDL invalidity condition
4. Vector spaces
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an algebra with easy direct limits	=	an algebra from which we can obtain by a direct limit construction a retract of itself only

$A = (A, F)$ algebra
$B = (B, F)$ subalgebra of A
B is said to be a retract of A if there exists an endomorphism φ of A such that

- $\varphi(A) = B$
- $\varphi(b) = b$ for every $b \in B$

Theorem (Laradji, 2002)

B is a retract of A iff every system of equations over B with a solution in A has a solution in B

Definition

A direct system of algebras $\{I, A_i, \varphi_{ij}\}$ contains

1. upward directed poset $\langle I, \leq \rangle$, $I \neq \emptyset$;
2. algebra (A_i, F) for each $i \in I$;
3. homomorphism φ_{ij} of A_i into A_j $(i < j)$;
   φ_{ii} the identity on A_i;
   $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ $(i < j < k)$.

Put $x \equiv y$ if $\varphi_{ik}(x) = \varphi_{jk}(y)$

The direct limit of $\{I, A_i, \varphi_{ij}\}$ is $(\overline{A}, F)$, where

- $\overline{A} = \dot{\bigcup}_{i \in I} A_i / \equiv$
- $f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) = \overline{f(\varphi_{i_1 k}(x_1), \ldots, \varphi_{i_n k}(x_k))}$ for n-ary $f \in F$

(A, F) ...... an algebra

$\underrightarrow{\mathbf{L}}(A, F)$ ... the class of all isomorphic copies of direct limits which can be obtained from A

$\mathbf{R}(A, F)$ ... the class of all isomorphic copies of retracts of (A, F)

$[(A, F)]$ ... the class of all isomorphic copies of (A, F)

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$[(A, F)]$ ... the class of all isomorphic copies of (A, F)

Lemma

$$\mathbf{R}(A, F) \subseteq \underrightarrow{\mathbf{L}}(A, F)$$

an algebra with easy direct limits	=	an algebra from which we can obtain by a direct limit construction a retract of itself only

We say that (A, F) is an algebra with easy direct limits (EDL) if

$$\underrightarrow{\mathbf{L}}(A, F) = \mathbf{R}(A, F).$$

Are basic algebraic structures on integer, rational, real and complex numbers algebras with EDL?

	ring	additive group	multiplicative monoid
$\mathbb{Z}$			
$\mathbb{Q}$			
$\mathbb{R}$			
$\mathbb{C}$			

In general

(A, F) algebra

Theorem (H., Ploščica 1999)

If A is finite, then (A, F) is with EDL.

Lemma

If every non-constant endomorphism of (A, F) is an automorphism, then (A, F) is with EDL.

Are basic algebraic structures on integer, rational, real and complex numbers algebras with EDL?

	ring	additive group	multiplicative monoid
$\mathbb{Z}$	$\surd$		
$\mathbb{Q}$	$\surd$	$\surd$	
$\mathbb{R}$	$\surd$		
$\mathbb{C}$			

Denote by $\mathcal{P}_{(A,F)}$ the property that
there exists $(B, F) \in \underrightarrow{\mathbf{L}}(A, F)$ such that $\|B\| > \|A\|$.

Proposition

Let (A, F) be an algebra such that $\mathcal{P}_{(A,F)}$ is valid. If $G \subseteq F$, then the algebra (A, G) is not with EDL.

Proposition

Let A_m be an algebra of type F which has a constant endomorphism e_m for each $m \in M$. Denote

$$A = \prod_{m \in M} A_m.$$

If $k \in M$ is such that $\|A_k\| = \|A\|$ and $\mathcal{P}_{A_k}$ is valid, then A is not with EDL.

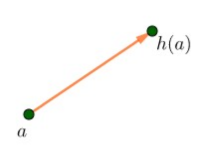
Are basic algebraic structures on integer, rational, real and complex numbers algebras with EDL?

	ring	additive group	multiplicative monoid
$\mathbb{Z}$	$\surd$		
$\mathbb{Q}$	$\surd$	$\surd$	
$\mathbb{R}$	$\surd$		**X if** $\mathcal{P}_{\text{addit.group } \mathbb{R}}$
$\mathbb{C}$		**X if** $\mathcal{P}_{\text{ring } \mathbb{C}}$	**X if** $\mathcal{P}_{\text{ring } \mathbb{C}}$

A mono-unary EDL invalidity condition

$A \neq \emptyset$, $h : A \to A$
(A, h) monounary algebra

$a \in A$



cyclic element
cycle
connected algebra
component
source

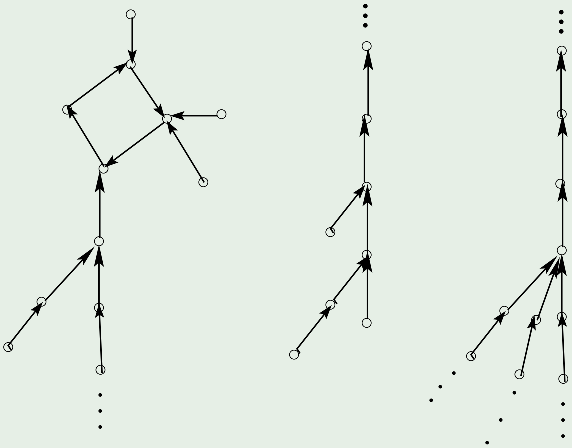
$\sum_{i\in I}(B_i, h)$ denotes a monounary algebra which is a disjoint union of algebras (B_i, h), $i \in I$.
Let $(A, h) = \sum_{i\in I}(B_i, h)$ and (B_i, h) be connected for all $i \in I$.
If (B_i, h) contains a cycle of length k, $k \in \mathbb{N}$, then we take (C_i, h) a cycle of length k. Else we take (C_i, h) a line. Put

$$(A, h)^{\diamond} = \sum_{i\in I}(C_i, h).$$

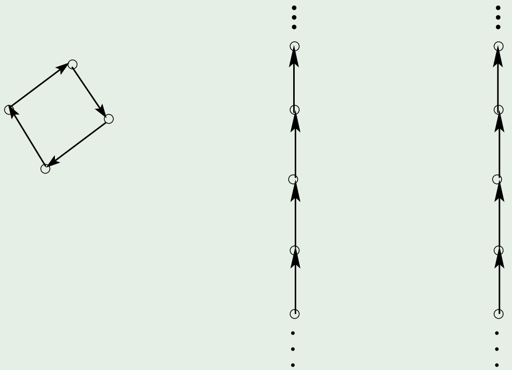
Lemma

$(A, h)^{\diamond} \in \underset{\rightarrow}{\mathbf{L}}(A, h)$

Example: (A, h)



Example: $(A, h)^\diamond$



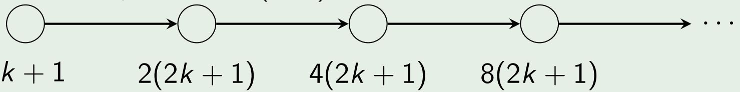
Theorem

Let (A, F) be an algebra and h be an unary term operation over F such that h is an endomorphism of the algebra (A, F).
If (A, F) is with EDL, then $(A, h)^{\diamond} \in \mathbf{R}(A, h)$.

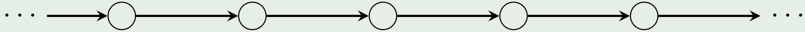
Additive group of $\mathbb{Z}$

$h(x) = 2x$

Infinite components of $(\mathbb{Z}, h)$, $k \in \mathbb{Z}$



Infinite components of $(\mathbb{Z}, h)^\diamond$



Multiplicative monoids of $\mathbb{Z}, \mathbb{Q}$

Use $g(x) = x^2$

Are basic algebraic structures on integer and rational numbers algebras with EDL?

	ring	additive group	multiplicative monoid
$\mathbb{Z}$	$\surd$	X	X
$\mathbb{Q}$	$\surd$	$\surd$	X

Vector spaces

Lemma

Let $\varphi : \mathbb{R} \to \mathbb{R}$. Then the following properties are equivalent:

1. $\varphi(a+b) = \varphi(a) + \varphi(b)$ *for each* $a, b \in \mathbb{R}$,
2. φ *is an endomorphism of the additive group of* $\mathbb{R}$,
3. φ *is an endomorphism of the vector space* $\mathbb{R}$ *over* $\mathbb{Q}$.

Corollary

The group $(\mathbb{R}, +, -, 0)$ *has EDL if and only if the vector space* $\mathbb{R}$ *over* $\mathbb{Q}$ *has EDL.*

Lemma

Let V be a vector space over F and $W \subseteq V$. TFAE:

1. *W is a retract of V,*
2. *W is a vector space.*

Theorem

Let V be a vector space. TFAE

1. *the dimension of V is finite,*
2. *V is with EDL.*

Let V be over a field K and its dimension be infinite.
Then there exists a Hamel basis of V, i.e.,
$H = \{h_t, t \in T\}$ such that

V is generated by H and
H is linearly independent.

V is generated by H if for every $v \in V$ there exist uniquely determined $a_t \in K, t \in T$ such that

- $v = \sum_{t \in T} a_t h_t$ and
- $a_t \neq 0$ for finitely many indexes.

H is linearly independent means that
if $n \in \mathbb{N}$ and $\sum_{k=1}^{n} a_k h'_k = 0$, then $a_k = 0$ for each $k \in \{1, ..., n\}$.

$H = \{h_t, t \in T\}$, Hamel basis of V
Take $t' \in T$. Suppose that

$$\psi : H \to H \setminus \{h_{t'}\}$$

is a bijective mapping.
We denote by κ the smallest ordinal whose cardinality is greater than $\|V\|$.
We built by a transfinite induction an V-uniform direct family of vector spaces $\{\kappa, A_i, \varphi_{i,j}\}$ where $\varphi_{i,i+1}$ coincide with ψ.
Elements $h_{t'}$ create new elements in the direct limit and therefore we obtain an algebra of cardinality greater than $\|V\|$.

Are basic algebraic structures on integer, rational, real and complex numbers algebras with EDL?

	ring	additive group	multiplicative monoid
$\mathbb{Z}$	$\surd$	X	X
$\mathbb{Q}$	$\surd$	$\surd$	X
$\mathbb{R}$	$\surd$	X	X
$\mathbb{C}$		**X if** $\mathcal{P}_{\mathrm{ring}\ \mathbb{C}}$	**X if** $\mathcal{P}_{\mathrm{ring}\ \mathbb{C}}$

Simple algebras

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Let A be a simple algebra.

Lemma

If $B \in \underrightarrow{\mathbf{L}}(A)$, then B is simple.

Lemma

If every injective endomorphism of A is surjective, then A is with EDL.

Theorem

Let every non-constant operation of A be unary. Then A is with EDL.

Theorem

Let A be a simple algebra such that there exists an injective endomorphism of A which is not surjective.
Then A is not with EDL.

$\mathbb{R}(A) \subseteq [A, \{a\}]$
Proof works similarly (not analogously!) as for vector spaces of infinite dimension.

Are basic algebraic structures on integer, rational, real and complex numbers algebras with EDL?

	ring	additive group	multiplicative monoid
$\mathbb{Z}$	$\surd$	X	X
$\mathbb{Q}$	$\surd$	$\surd$	X
$\mathbb{R}$	$\surd$	X	X
$\mathbb{C}$	X	X	X

Conclusion

Summary

- vector spaces with EDL are exactly finite dimensional ones
- finitely generated abelian groups with EDL are exactly finite ones
- simple algebras with EDL are exactly those that have every non-constant endomorphism bijective
- several other classes of algebras with EDL are described
- every monounary algebra with EDL is countable and it does not hold generally
- a monounary EDL invalidity condition works for some algebras

Questions

- Describe classes of direct limits of algebras from the table that are not with EDL.
- Is there an algebra with EDL of cardinality greater that continuum?
- Is there an algebra with EDL which has a retract which is not with EDL?

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