Kőnig = Ramsey, A compactness lemma for Ramsey categories

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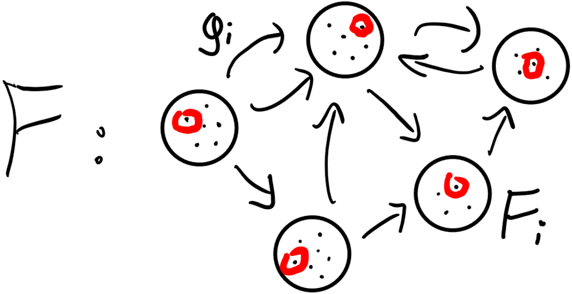
Solving diagrams

INPUT: a *diagram*: some finite nonempty sets and some maps between them



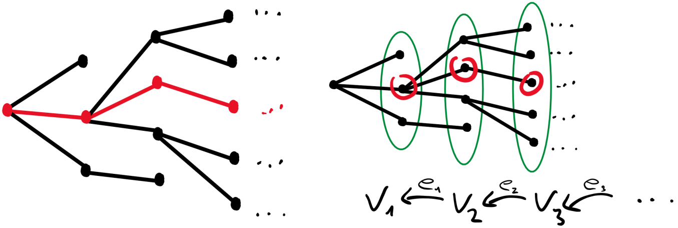
TASK: Pick one element out of each set, compatibly with all maps

$$\{\text{solutions}\} = \{(x_i \in \mathcal{F}_i)_i \mid \forall g_i : \mathcal{F}_h \to \mathcal{F}_k : g(x_h) = x_k\}$$

Kőnig's tree lemma

Lemma (Kőnig, 1927)

Any finitely branching infinite tree contains an infinite path.



infinite paths $\Longleftrightarrow$ solutions

Kőnig's Lemma (rephrased)

Every diagram in the shape of $(\mathbb{N}, \leq)^{\mathrm{op}}$ has a solution.

Example: 3-colouring

Example:

A *countable* graph G is 3-colourable if all its finite subgraphs are 3-colourable.

Proof.

If G countable:

finite

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G$$

finite non-empty

$$\mathcal{F} : (\mathbb{N}, \leq)^{\mathrm{op}} \to \mathrm{Set}, \ n \mapsto \mathrm{Hom}(G_n, K_3)$$

Kőnig

$$\lim \mathcal{F} = \mathrm{Hom}(G, K_3)$$

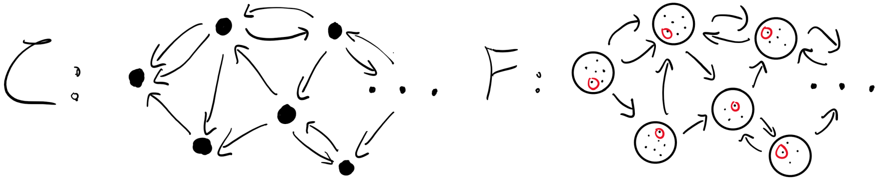
non-empty

□

Improvements?

Lemma (Kőnig, 1927), rephrased

Every diagram in the shape of $(\mathbb{N}, \leq)^{\mathrm{op}}$ has a solution.



Definition

Call a category $\mathcal{C}$ *Kőnig* if every diagram in the shape of $\mathcal{C}^{\mathrm{op}}$ has a solution.

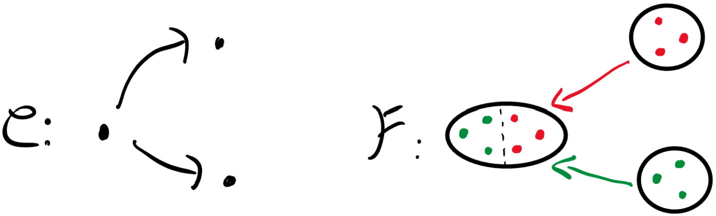
Theorem (H.)

For a small, locally finite category $\mathcal{C}$, TFAE:

1. $\mathcal{C}$ is confluent and Ramsey
2. $\mathcal{C}$ is Kőnig, i.e. every diagram in the shape of $\mathcal{C}^{\mathrm{op}}$ has a solution

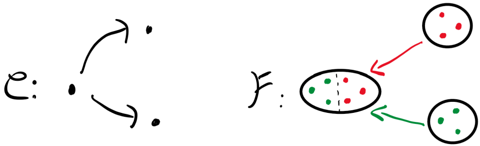
Obstacles for being Kőnig

Forks



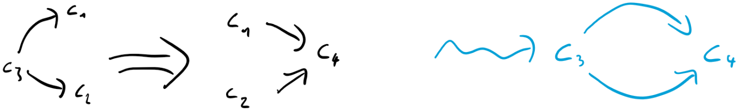
Parallel arrows

Obstacles: Forks



Definition

A category is called *confluent* if every two objects with a common lower bound also have a common upper bound.



Solving parallel arrows: ignorance

Lemma

Posets (no parallel arrows) are Kőnig if and only if they are confluent.

Solving parallel arrows: ignorance

Lemma

Posets (no parallel arrows) are Kőnig if and only if they are confluent.

Example

A graph G is 3-colourable if all its finite subgraphs are 3-colourable.

Proof.

confluent

$$\mathcal{C} := (\text{all finite subgraphs of } G, \text{ordered by inclusion})$$

$G_1 \subseteq G_1 \cup G_2$, $G_2 \subseteq G_1 \cup G_2$

$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set},\ (G_i \subseteq G) \mapsto \mathrm{Hom}(G_i, K_3)$$

finite non-empty

Kőnig

$$\lim \mathcal{F} = \mathrm{Hom}(G, K_3)$$

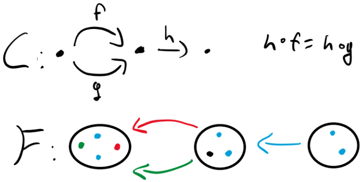
non-empty

□

Solving parallel arrows: cheating

Potential fix

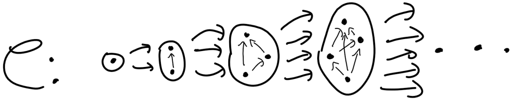
Require that for any two arrows f, g there is h "equalizing" them.



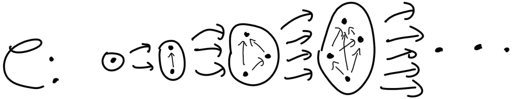
Proposition

this + confluent $\implies$ Kőnig

Solving parallel arrows: Ramsey

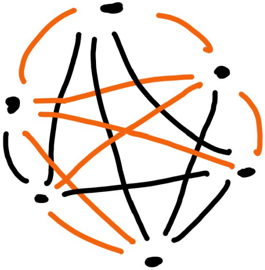


Solving parallel arrows: Ramsey



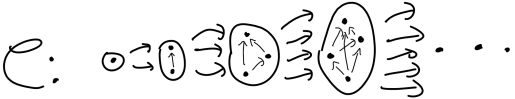
Theorem (Ramsey 1928)

For any k there exists n such that any 2-edge coloring of the complete n element graph contains a monochromatic subgraph of size k.



$k = 3, n = 6$

Solving parallel arrows: Ramsey



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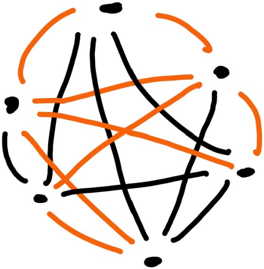

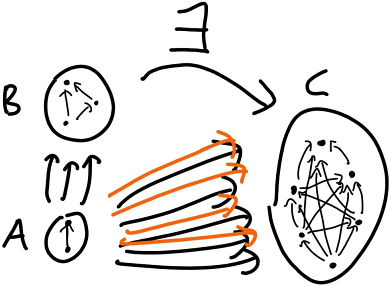

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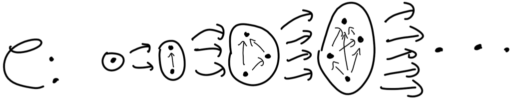
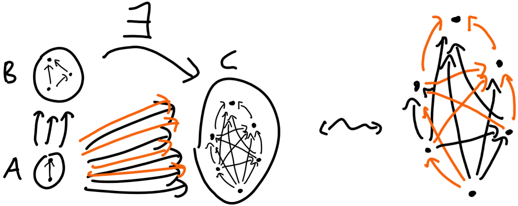
Solving parallel arrows: Ramsey



Morally

For any colouring of arrows, there is h such that $h \circ f$ and $h \circ g$ have the same colour.



The Ramsey Property

Morally

For any colouring of arrows, there is h such that $h \circ f$ and $h \circ g$ have the same colour.

Definition

A category $\mathcal{C}$ is called *Ramsey* if for all $A, B \in \mathcal{C}$, there is $C \in \mathcal{C}$ such that for all $\chi : \mathrm{Hom}(A, C) \to \{0, 1\}$ there is $h : B \to C$ such that

$$\mathrm{Hom}(A, B) \xrightarrow{h_*} \mathrm{Hom}(A, C) \xrightarrow{\chi} \{0, 1\}$$

is constant.

This is enough to fix parallel arrows!

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$=$

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Whats the point?

Corollaries

- improved canonization lemma (infinite CSPs)
- understanding minimal Ramsey expansions
- new Ramsey transfers

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Fun Fact

Every locally finite, infinite-dimensional simplicial set contains the infinite dunce cap (the simplicial set with precisely one nondegenerate simplex of each dimension).

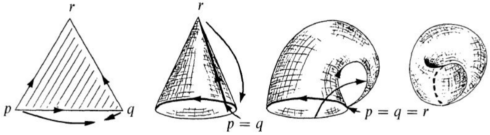

Figure I-6. The dunce cap.