Amalgamation in lattice-ordered groups and cancellative residuated structures

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Back to cancellativity

Recall that a residuated lattice is cancellative if both of

$$xz = yz \Rightarrow x = y, \text{ and}$$

$$zx = zy \Rightarrow x = y$$

hold, or, equationally,

$$(xy)/y = x = y\backslash(yx).$$

Lattice-ordered groups are examples of cancellative RLs, but it turns out this class is much wider.

Back to cancellativity

Recall that ℓ-groups are always distributive as lattices. In contrast:

Theorem (Bahls, Cole, Galatos, Jipsen, Tsinakis, 2003):

Every lattice appears as a sublattice of a cancellative, integral residuated lattice. Indeed, there is an order-preserving injection of the lattice of lattice varieties into the lattice of subvarieties of cancellative residuated lattices.

Some guiding examples

As the previous theorem suggests, quite a lot can happen for cancellative residuated lattices that does not happen for ℓ-groups.

There are integral examples: If $\mathbf{A}$ is any ℓ-group, then we may define a cancellative residuated lattice by taking the negative cone of $\mathbf{A}$,

$$A^- = \{a \in A \mid a \leq e\},$$

with the inherited order and product. The residuals are given by:

$$x\backslash y = (x^{-1}y) \wedge e, \quad y/x = (yx^{-1}) \wedge e.$$

Thus both ℓ-groups and negative cones of ℓ-groups give examples of cancellative RLs.

Some guiding examples

...but the class of cancellative RLs is bigger still. Take $\mathbf{M}$ to be the free 2-generated monoid with free generators a and b. It is observed in [Bahl-Cole-Galatos-Jipsen-Tsinakis 2003] that $\mathbf{M}$ can be totally ordered by stipulating that $u \leq v$ if and only if the length of u is greater than the length of v, or u and v have the same length but $u < v$ in the lexicographic order given by $b < a$:

$$\cdots b^3 < ab^2 < a^2b < a^3 < b^2 < ab < a^2 < b < a < e.$$

This is a cancellative, commutative, integral, totally ordered residuated lattice. However, $(b/a)a = a^2$ and $b = a \wedge b$ in $\mathbf{M}$, but the identity $(x/y)y = x \wedge y$ holds in all negative cones of ℓ-groups.

Divisibility

A residuated lattice is called divisible if it satisfies

$$y \leq x \implies x(\backslash y) = y = (y/x)x.$$

This is equivalent to the pair of identities

$$x[x\backslash(x \wedge y)] = x \wedge y = [(x \wedge y)/x]x,$$

which, in the presence of integrality, is

$$x(x\backslash y) = x \wedge y = (y/x)x.$$

Divisible residuated lattices are called GBL-algebras (more on the name later) and they have been very important for understanding the connection between ℓ-groups and more general RLs.

Theorem (Galatos-Tsinakis 2005):

1. GBL-algebras are exactly those residuated lattices isomorphic to algebras of the form $\mathbf{A} \times \mathbf{B}$, where $\mathbf{A}$ is an ℓ-group and $\mathbf{B}$ is an integral GBL-algebra.
2. In particular, cancellative GBL-algebras are exactly those algebras isomorphic to algebras of the form $\mathbf{A} \times \mathbf{B}$, where $\mathbf{A}$ is an ℓ-group and $\mathbf{B}$ is the negative cone of an ℓ-group.

Amalgamation beyond ℓ-groups

The following is an immediate corollary of the preceding structure theorem.

Proposition (Metcalfe-Montagna-Tsinakis 2014):

Let V be any variety of GBL-algebras that contains all ℓ-groups. Then V does not have the AP. In particular, neither the variety of all GBL-algebras nor the variety of all cancellative GBL-algebras has the AP.

Unlike ℓ-groups, however, there are non-semilinear subvarieties of GBL-algebras with the AP: the subvariety of 0-free subreducts of Heyting algebras, for example.

ℓ-groups and their negative cones

Theorem (Bahls-Cole-Galatos-Jipsen-Tsinakis 2003):

1. The variety of ℓ-groups is categorically equivalent to the variety of negative cones of ℓ-groups ($=$ cancellative and integral GBL-algebras).
2. The variety of abelian ℓ-groups is categorically equivalent to the variety of negative cones of abelian ℓ-groups.

It follows immediately from (1) that the variety of cancellative and integral GBL-algebras does not have the AP, whereas from (2) the variety of negative cones of abelian ℓ-groups does have the AP. This also works to show that the variety of negative cones of representable ℓ-groups fails the AP.

Some other varieties

Theorem (Gil-Férez, Ledda, Tsinakis, 2015):
The variety of cancellative semilinear residuated lattices does not have the AP.

We do not know whether the following varieties have the AP:

- the variety of cancellative commutative residuated lattices
- the variety of semilinear cancellative commutative RLs
- the variety of integral cancellative semilinear residuated lattices

There are many other interesting subvarieties of GBL-algebras, often with close ties to ℓ-groups.

- A basic hoop is an integral, commutative, semilinear GBL-algebra.
- A BL-algebra is a basic hoop with a (designated) least element.
- A Wajsberg hoop is a basic hoop that satisfies $(x \to y) \to y = x \vee y$.
- An MV-algebra is a Wajsberg hoop with a (designated) least element.

Varieties derived from ℓ-groups

MV-algebras (and therefore Wajsberg hoops) famously come from intervals of abelian ℓ-groups by applying the Mundici functor.

[Aglianò-Montagna 2003] shows that totally ordered BL-algebras are all ordinal sums, made from stacking up Wajsberg hoops and MV-algebras.

BL-algebras are also exactly the algebras generated by continuous t-norms on $[0, 1]$.

Amalgamation around MV/Wajsberg

Di Nola-Lettieri 2000: A variety of MV-algebras has the AP if and only if it is generated by a single totally ordered MV-algebra.

Metcalfe-Montagna-Tsinakis 2014: Almost true of Wajsberg hoops as well, but can also include $\mathbb{Z}^-$.

Amalgamation around BL

Quite a history of trying to pin down a classification of subvarieties of BL-algebras with the AP:

[Montagna 2006]: Variety of all BL-algebras + many of the most natural subvarieties have AP, but there are uncountably many that do not. Very few extensions of basic logic with CIP.

Montagna's problem: How many varieties of BL-algebras have AP? Countably many or uncountably many? Structure of $\Omega(\mathrm{BL})$?

[Cortonesi-Marchioni-Montagna 2011]: Applied tools from first-order model theory.

[Aguzzoli-Bianchi 2021]: Partial classification for finitely generated varieties.

[Fussner-Metcalfe 2022]: New general results for studying AP.

[Aguzzoli-Bianchi 2023]: Sharpened classification, but still not complete.

Some building blocks

We define some algebras:

- Ł$_n$ the n-element MV-algebra chain.
- **Z** the negative cone of the integers (a cancellative hoop)
- $\mathbf{W}_m$ the 0-free reduct of the MV-algebra Ł$_m$.
- $\mathbf{W}_{m,\omega}$ the 0-free reduct of the MV-algebra $\Gamma(\mathbb{Z} \times \mathbb{Z}, \langle m, 0 \rangle)$, where $\mathbb{Z} \times \mathbb{Z}$ is ordered lexicographically as an ℓ-group and Γ is the Mundici functor.

Useful nomenclature

We introduce some naming conventions for varieties:

- $\mathbf{A} \oplus \mathbf{B}$ is written $\mathbf{AB}$
- The class generated by the componentwise $\mathrm{HSP_u}$ closure of an ordinal sum by enclosing the corresponding ordinal sum in bracket [,], so that, for example, $[\mathbf{AB}]$ denotes the class of all ordinal sums $\mathbf{A}' \oplus \mathbf{B}'$ where $\mathbf{A}' \in \mathrm{HSP_u}(\mathbf{A})$ and $\mathbf{B}' \in \mathrm{HSP_u}(\mathbf{B})$; and $[\mathbf{A}] = \mathrm{HSP_u}(\mathbf{A})$.
- We use * to denote the repetition of one or more instances of a summand in a given ordinal sum. For example, $[\mathbf{AB}^*]$ abbreviates the class consisting of all ordinal sums of the form $\mathbf{A} \oplus \mathbf{B}_1 \oplus \cdots \oplus \mathbf{B}_n$, where n is a positive integer and $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathrm{HSP_u}(\mathbf{B})$.
- Kleene star * has priority over $\oplus$, so that $[\mathbf{ABC}^*]$ abbreviates $[(\mathbf{A} \oplus \mathbf{B}) \oplus \mathbf{C}^*]$.

A decomposition into intervals

For each variety V of basic hoops, denote by $\mathit{Wajs}(\mathsf{V})$ the class of Wajsberg chains in V. If $\mathsf{V} \in \Omega(\mathsf{BH})$, then $\mathbb{V}(\mathit{Wajs}(\mathsf{V})) \in \Omega(\mathsf{WH})$.

Theorem (F.-Santschi 2025):

The poset $\Omega(\mathsf{BH})$ can be partitioned into countably infinitely many closed intervals: for any variety V of basic hoops with the amalgamation property one of the following holds.

1. V is trivial.
2. $\mathrm{Wajs}(\mathsf{V}) = [\mathbf{A}]$ for $\mathbf{A} \in \{\mathbf{W}_n \mid n \geq 1\} \cup \{\mathbf{Z}, [0,1]_{\mathsf{WH}}\}$, and $[\mathbf{A}] \subseteq \mathsf{V}_{\mathrm{fc}} \subseteq [\mathbf{A}^*]$.
3. $\mathrm{Wajs}(\mathsf{V}) = [\mathbf{W}_{n,\omega}]$ for some $n \geq 1$, and $[\mathbf{W}_{n,\omega}] \subseteq \mathsf{V}_{\mathrm{fc}} \subseteq [\mathbf{W}^*_{n,\omega}]$.
4. $\mathrm{Wajs}(\mathsf{V}) = [\mathbf{W}_n] \cup [\mathbf{Z}]$ for some $n \geq 1$, and $[\mathbf{W}_n] \cup [\mathbf{Z}] \subseteq \mathsf{V}_{\mathrm{fc}} \subseteq [(\mathbf{W}_n\mathbf{Z})^*]$.

Closure properties

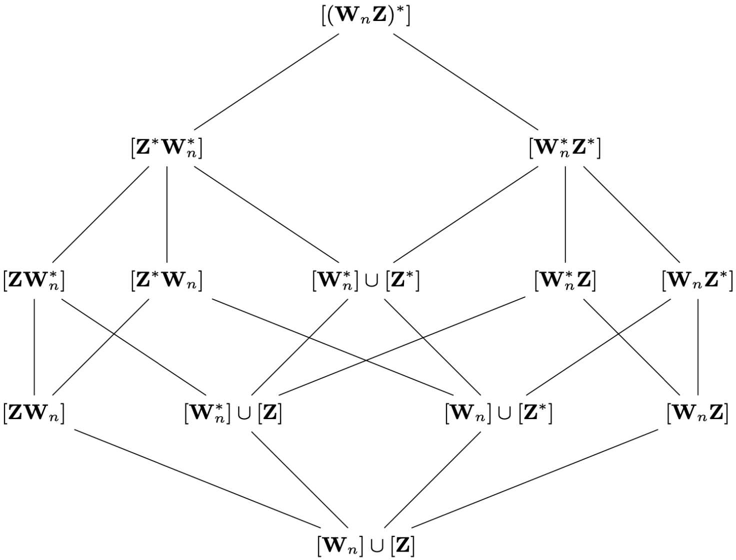
Lemma (F.-Santschi 2025):

Let $\mathsf{V} \in \Omega(\mathsf{BH})$. If $\bigoplus_{i=1}^{l} \mathbf{A}_i \in \mathsf{V}_{\mathrm{fc}}$ with $\mathbf{A}_i \in \mathrm{Wajs}(\mathsf{V})$, then $\bigoplus_{i=1}^{l} \mathrm{H}(\mathbf{A}_i) \subseteq \mathsf{V}_{\mathrm{fc}}$.

Lemma (F.-Santschi 2025):

Let $\mathsf{V} \in \Omega(\mathsf{BH})$ and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ Wajsberg chains.

1. If $\mathbf{A} \oplus \mathbf{B} \oplus \mathbf{B} \oplus \mathbf{C} \in \mathsf{V}_{\mathrm{fc}}$, then $[\mathbf{A}\mathbf{B}^*\mathbf{C}] \subseteq \mathsf{V}_{\mathrm{fc}}$.
2. If $\mathbf{W}_n \oplus \mathbf{W}_{n,\omega} \in \mathsf{V}_{\mathrm{fc}}$, then $[\mathbf{W}_n^*\mathbf{W}_{n,\omega}] \subseteq \mathsf{V}_{\mathrm{fc}}$.
3. If $\mathbf{A} \oplus \mathbf{A} \oplus \mathbf{B} \oplus \mathbf{B} \in \mathsf{V}_{\mathrm{fc}}$ or $\mathbf{A} \oplus \mathbf{A}, \mathbf{A} \oplus \mathbf{B}, \mathbf{B} \oplus \mathbf{B} \in \mathsf{V}_{\mathrm{fc}}$, then $[\mathbf{A}^*\mathbf{B}^*] \subseteq \mathsf{V}_{\mathrm{fc}}$.
4. If $\mathbf{A} \oplus \mathbf{B}, \mathbf{B} \oplus \mathbf{A} \in \mathsf{V}_{\mathrm{fc}}$, then $[(\mathbf{A}\mathbf{B})^*] \subseteq \mathsf{V}_{\mathrm{fc}}$.


$[(\mathbf{W}_n\mathbf{Z})^*]$
$[\mathbf{Z}^*\mathbf{W}_n^*]$
$[\mathbf{W}_n^*\mathbf{Z}^*]$
$[\mathbf{Z}\mathbf{W}_n^*]$
$[\mathbf{Z}^*\mathbf{W}_n]$
$[\mathbf{W}_n^*] \cup [\mathbf{Z}^*]$
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$[\mathbf{Z}\mathbf{W}_n]$
$[\mathbf{W}_n^*] \cup [\mathbf{Z}]$
$[\mathbf{W}_n] \cup [\mathbf{Z}^*]$
$[\mathbf{W}_n\mathbf{Z}]$
$[\mathbf{W}_n] \cup [\mathbf{Z}]$

Taking stock

Theorem (F.-Santschi 2025):

The poset $\Omega(\mathrm{BH})$ can be partitioned into countably many finite intervals.

Closure properties for BL

Lemma (F.-Santschi 2025):

Let $\mathsf{V} \in \Omega(\mathsf{BL})$ be such that $\mathbf{A}_0 \oplus \mathbf{A}_1 \in \mathsf{V}_{\mathrm{fc}}$, where $\mathbf{A}_0$ is a non-trivial totally ordered MV-algebra and $\mathbf{A}_1$ a totally ordered basic hoop.

1. If $\mathbf{B}_0 \in Ł(\mathsf{V})$ and $\mathbf{A}_0 \leq \mathbf{B}_0$ is an essential extension, then $\mathbf{B}_0 \oplus \mathbf{A}_1 \in \mathsf{V}_{\mathrm{fc}}$.
2. If $\mathbf{B}_0 \in Ł(\mathsf{V})$ is simple, then $\mathbf{B}_0 \oplus \mathbf{A}_1 \in \mathsf{V}_{\mathrm{fc}}$.
3. If for $m \geq 1$, $\mathbf{Ł}_{m,\omega} \in Ł(\mathsf{V})$ and $\mathbf{A}_0 \in [\mathbf{Ł}_{m,\omega}] \setminus [\mathbf{Ł}_m]$, then $\mathbf{Ł}_{m,\omega} \oplus \mathbf{A}_1 \in \mathsf{V}_{\mathrm{fc}}$.
4. If $\mathbf{B}_0 \oplus \mathbf{B}_1 \in \mathsf{V}_{\mathrm{fc}}$ such that $\mathbf{A}_0 \leq \mathbf{B}_0$, $\mathbf{B}_1 \leq \mathbf{A}_1$, and $\mathbf{B}_1$ is non-trivial, then $\mathbf{B}_0 \oplus \mathbf{A}_1 \in \mathsf{V}_{\mathrm{fc}}$.
5. If $\mathbf{B}_0 \oplus \mathbf{B}_1 \in \mathsf{V}_{\mathrm{fc}}$ such that $\mathbf{A}_1 \leq \mathbf{B}_1$ is an essential extension, then $\mathbf{A}_0 \oplus \mathbf{B}_1 \in \mathsf{V}_{\mathrm{fc}}$.

Intervals for BL

For a variety V of BL-algebras, let $\mathrm{Basic_{fc}}(\mathsf{V})$ be the class of finite index basic hoops appearing in ordinal sum decompositions in chains in V.

Theorem (F.-Santschi 2025):

Let V be a variety of BL-algebras. Then $\mathsf{V} \in \Omega(\mathsf{BL})$ if and only if $\mathbb{V}(\mathrm{Basic_{fc}}(\mathsf{V})) \in \Omega(\mathsf{BH})$ and one of the following holds:

1. V is trivial.
2. $Ł(\mathsf{V}) = [\mathbf{A}]$ for some $\mathbf{A} \in \{\mathbf{Ł}_m, \mathbf{Ł}_{m,\omega} \mid m \in \mathbb{N}\} \cup \{[0,1]_{\mathsf{MV}}\}$ and $\mathsf{V}_{\mathrm{fc}} = [\mathbf{A}] \oplus \mathrm{Basic_{fc}}(\mathsf{V})$.
3. $Ł(\mathsf{V}) = [\mathbf{Ł}_{m,\omega}]$ for some $m \geq 1$ and $\mathsf{V}_{\mathrm{fc}} = ([\mathbf{Ł}_m] \oplus \mathrm{Basic_{fc}}(\mathsf{V})) \cup [\mathbf{Ł}_{m,\omega}]$.
4. $Ł(\mathsf{V}) = [\mathbf{Ł}_{m,\omega}]$ for some $m \geq 1$, $\mathrm{Basic_{fc}}(\mathsf{V}) = \mathsf{K}_1 \cup \mathsf{K}_2$, where $\mathsf{K}_1 \in \{[\mathbf{W}_n], [\mathbf{W}_n^*]\}$ for some $n \geq 1$ and $\mathsf{K}_2 \in \{[\mathbf{Z}], [\mathbf{Z}^*]\}$; and either $\mathsf{V}_{\mathrm{fc}} = ([\mathbf{Ł}_{m,\omega}] \oplus \mathsf{K}_1) \cup ([\mathbf{Ł}_m] \oplus \mathsf{K}_2)$, or $\mathsf{V}_{\mathrm{fc}} = ([\mathbf{Ł}_m] \oplus \mathsf{K}_1) \cup ([\mathbf{Ł}_{m,\omega}] \oplus \mathsf{K}_2)$.

The main result for BL

Theorem (F.-Santschi 2025):

The poset $\Omega(\mathrm{BL})$ can be partitioned into countably many finite intervals.

Conclusion

There remain many open problems around AP.

- Does any non-abelian variety of ℓ-groups have AP?
- A range of even murkier problems around cancellative RLs.

However, there has been a lot of progress on AP in recent years.

- The solution of Montagna's problem required very new techniques.
- Progress on varieties without CEP, important for non-commutative varieties of RLs.

Thank you!