Amalgamation in lattice-ordered groups and cancellative residuated structures

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Amalgamation in ℓ-groups: the landscape

Recall that an ℓ-group is representable (or semilinear) if it is a subdirect product of linearly ordered ℓ-groups. We know:

- Abelian ℓ-groups are representable.
- Abelian ℓ-groups have the AP.

Today we will see:

- Gurchenkov's theorem: Every variety with the AP is representable.
- Among representable ℓ-groups, the varieties that may have the AP are rather restricted.
- In particular, the variety of all ℓ-groups lacks the AP.

The entire variety

Theorem (Pierce 1973):

The variety of all ℓ-groups lacks the AP.

Lexicographic products

We follow a proof of Metcalfe, Paoli, and Tsinakis (2023). For this, we consider the lexicographic product.

Given totally ordered ℓ-groups **A** and **B**, we can turn the direct product of the underlying groups of **A** and **B** into a totally ordered ℓ-group by imposing the lexicographic order:

$$(a, b) \leq (a', b') \iff a < a', \text{ or } a = a' \text{ and } b \leq b'.$$

Of course, we could also extend this to arbitrary numbers of factors as well as take the dual lexicographic product if we wished.

Hahn embedding theorem: Every totally ordered abelian group embeds in a lex product of copies of $\mathbb{R}$.

ℓ-groups do not have AP

Lex products are important in constructing many examples. For instance, to see that the variety of all ℓ-groups LG does not have AP, let $\mathrm{Aut}(\mathbb{Z} \times \mathbb{Z})$ be the ℓ-group of order-preserving permutations of $\mathbb{Z} \times \mathbb{Z}$ (ordered as a direct product).

Consider a (group) homomorphism $h\colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z} \times \mathbb{Z})$ defined by $h(n)(i, j) = (i, j)$ if n is even and $h(n)(i, j) = (j, i)$ if n is odd.

We let $\mathbf{B}$ be the semidirect product $(\mathbb{Z} \times \mathbb{Z}) \rtimes_h \mathbb{Z}$ ordered dually lexicographically.

Take $\mathbf{A}$ to be the subalgebra of $\mathbf{B}$ generated by $(0, 0, 1)$, $(0, 1, 0)$, and $(0, 0, 1)^2$.

Also, consider $\mathbf{C}_1$ to be $\mathbb{Z} \times \mathbb{Z}$ ordered dually lexicographically, set $\mathbf{C}_2 = \mathbf{C}_1 \times \mathbf{C}_1$, and let $\mathbf{C}$ be the subalgebra of $\mathbf{C}_2$ generated by the unit vectors.

Now $\mathbf{A}$ is included in $\mathbf{B}$, and we may define an embedding g of $\mathbf{A}$ into $\mathbf{C}$ by setting $g(1,0,0) = (1,0,0,0)$, $g(0,1,0) = (0,0,1,0)$, and $g((0,0,1)^2) = (0,1,0,0)(0,0,0,1)$. One may show that the resulting span has no amalgam.

Representable ℓ-groups are enough

Theorem (Gurchenkov 1997):

Let V be any variety of ℓ-groups. If V has the AP, then V is representable.

Amalgamation bases

Let V, W be varieties of ℓ-groups with $\mathsf{V} \subseteq \mathsf{W}$. We say that an ℓ-group $\mathbf{A} \in \mathsf{V}$ is an amalgamation base for V in W if any span of the form $(f\colon \mathbf{A} \to \mathbf{B}, g\colon \mathbf{A} \to \mathbf{C})$ in V has an amalgam in W.

By $\mathrm{amal}_{\mathsf{W}}(\mathsf{V})$ we mean the class of all amalgamation bases for V in W, and by $\mathrm{amal}(\mathsf{V})$ we mean $\mathrm{amal}_{\mathsf{V}}(\mathsf{V})$.

Conjugate-orthogonal rank

The key technical ingredient toward understanding Guchenkov's theorem is the notion of conjugate-orthogonal rank.

If **A** is any ℓ-group, we say that **A** has conjugate-orthogonal rank n and write $\mathrm{co}(\mathbf{A}) = n$ if:

1. There exist strictly positive elements $g, a \in A$ such that $a \wedge (g^{-i}ag^{i}) = e$ for each $i = 1, \ldots, n$, and
2. for each pair of strictly positive elements $x, y \in A$, if $x \wedge (y^{-i}xy^{i}) = e$ for each $i = 1, \ldots, n$, then $x \wedge (y^{-(n+1)}xy^{n+1}) \neq e$.

A variety V of ℓ-groups has conjugate-orthogonal rank n such that for each $\mathbf{A} \in$ V we have $\mathrm{co}(\mathbf{A}) \neq n$ and there exists $\mathbf{B} \in$ V with $\mathrm{co}(\mathbf{B}) = n$. The variety V has infinite conjugate-orthogonal rank if for each n there is $\mathbf{A} \in$ V with $\mathrm{co}(\mathbf{A}) \geq n$.

Some key lemmas

By a clever application of an ℓ-group construction called the lexicographic semidirect extension, Gurchenkov proves the following lemma:

Lemma (Gurchenkov 1997):

Let V be any variety of ℓ-groups with $\mathrm{co}(\mathsf{V}) = n \geq 1$. Then there is a span in V that cannot be amalgamated among all ℓ-groups.

Further, Gurchenkov proves the following lemma, which today would be packaged as a closure property:

Lemma (Gurchenkov 1997):

If $\mathrm{co}(\mathsf{V}) \geq 1$ and $\mathbb{Z} \in \mathrm{amal}(\mathsf{V})$, then $\mathrm{co}(\mathsf{V}) = \infty$.

Normal-valued ℓ-groups

Let $\mathbf{A}$ be any ℓ-group. We say that a convex subgroup $\mathbf{B}$ of $\mathbf{A}$ is regular if it is completely meet-irreducible in the lattice of all convex ℓ-subgroups of $\mathbf{A}$. A regular subgroup $\mathbf{B}$ of $\mathbf{A}$ hence has a cover $\mathbf{B}^*$ in the lattice of convex ℓ-subgroups of $\mathbf{A}$, and when $\mathbf{B}$ is normal in $\mathbf{B}^*$ we say that $\mathbf{B}$ is a normal value.

An ℓ-group $\mathbf{A}$ is normal valued if all of its regular convex ℓ-subgroups are normal values. Surprisingly, the class of N of all normal-valued ℓ-groups is a variety, and it turns out to be the greatest proper subvariety of ℓ-groups.

Lemma (Gurchenkov 1997):

Suppose that V is a variety of ℓ-groups with $\mathbb{Z} \in \mathrm{amal}(\mathsf{V})$, and such that V contains at least one non-representable ℓ-group. Then $\mathsf{N} \subseteq \mathsf{V}$.

To finish the proof, suppose V is a non-representable variety of ℓ-groups with the AP. Then certainly $\mathbb{Z} \in \mathrm{amal}(\mathsf{V})$. Then $\mathsf{N} \subseteq \mathsf{V}$ by the last lemma. The variety of normal valued ℓ-groups contains a subvariety W with finite conjugate-orthogonal rank. By the first lemma, W has a span that cannot be amalgamated among all ℓ-groups, and hence certainly not in V.

Representable ℓ-groups

Theorem (Glass, Saracino, Wood, 1984):

The variety of representable ℓ-groups does not have the AP.

Taking stock

We know a bit about $\Omega(\mathsf{LG})$, the poset of subvarieties of ℓ-groups with the AP:

- Its unique atom is the variety of Abelian ℓ-groups, which has AP.
- Its unique co-atom is the variety of normal-valued ℓ-groups, which doesn't have AP.
- Somewhere in the middle is the variety of representable ℓ-groups, also lacking the AP but containing any variety with the AP.

If there is another subvariety of LG with the AP, the natural place to look among the representable covers of Abelian ℓ-groups. As it turns out, Abelian ℓ-groups have both representable and non-representable covers.

Positive cones

To define the most pertinent varieties, we note that if **A** is an ℓ-group, then the order on **A** can be recovered from its positive cone,

$$A^+ = \{x \in A \mid e \leq x\},$$

since by residuation we always have

$$x \leq y \iff e \leq yx^{-1}.$$

One often defines the order on an ℓ-group by way of its positive cone.

Wreath products

If **A** and **B** are groups, recall that the wreath product of **A** and **B** can be defined by taking vectors of the form $((a_i), b)$, where (a_i) is a vector of elements of **A** indexed by $i \in B$ such that the support $\{i \in B \mid a_i \neq e\}$ is finite.

The group operation of the wreath product $\mathbf{A}\mathrm{wr}\mathbf{B}$ can be defined by

$$((a_i), b) \cdot ((a'_i), b') = ((c_i), bb'),$$

with $c_i = a_i a'_{ib}$.

Medvedev's varieties

If **A** and **B** are totally ordered groups, then their wreath product can be totally ordered in two ways.

For the first of these, the positive cone is defined by stipulating that $((a_i), b) > (e, e)$ if and only if $b > e$, or $b = e$ and $a_j > e$ in **A**, where $j = \max\{j \in B \mid a_j \neq e\}$.

The second order just takes $((a_i), b) > (e, e)$ ifif $b > e$, or $b = e$ and $a_j > e$ for $j = \min\{j \in B \mid a_j \neq e\}$.

Call $\mathbf{W}^+$ the ℓ-group with the first order, and $\mathbf{W}^-$ the one with the second order.

Medvedev's varieties

Medvedev's varieties are respectively $\mathsf{M}^+ = \mathsf{V}(\mathbf{W}^+)$ and $\mathsf{M}^- = \mathsf{V}(\mathbf{W}^-)$.

Proposition (Medvedev):

The varieties M^+ and M^- are representable covers of the variety of Abelian ℓ-groups.

The Powell-Tsinakis theorem

Theorem (Powell and Tsinakis, 1989):

No variety of ℓ-groups containing either of M^+ or M^- has the AP.

Proof of Powell-Tsinakis

The proof proceeds separately for each of M^+ and M^-.

Focusing on M^+, we want to find a span $(f\colon \mathbf{A} \to \mathbf{B}, g\colon \mathbf{A} \to \mathbf{C})$ that cannot be amalgamated among representable ℓ-grooups, hence in particular in M^+.

The idea of the proof uses the remarkable fact that representable ℓ-groups have unique roots for all exponents:

$$x^n = y^n \implies x = y.$$

Cyclic extensions

The correct ℓ-groups **A**, **B**, **C** can be found by considering cyclic extensions. If **A** is a totally ordered group and α is an order automorphism of **A**, then we define an ℓ-group $\mathbf{A}(\alpha)$ on the set $A \times \{\alpha^n \mid n \in \mathbb{Z}\}$ by

$$(a, \alpha^n) \cdot (b, \alpha^m) = (a\alpha^n(b), \alpha^{n+m}),$$

and ordered by $(a, \alpha^n) > (e, \alpha^0)$ if and only if either $n > 0$, or else $n = 0$ and $a > e$.

Concluding the proof

One can find the appropriate span by finding order automorphisms α, β, γ of totally ordered group **A** such that $\alpha = \beta^n = \gamma^n$. In this event, $\mathbf{A}(\alpha)$ embeds in each of $\mathbf{A}(\beta)$ and $\mathbf{A}(\gamma)$ and the automorphisms witness the failure of the AP.

Concretely, the ℓ-group **A** can be taken to be a large lexicographic product of copies of $\mathbb{Z}$.

The main technical challenge is in constructing the appropriate order automorphisms.

Consequences of the Powell-Tsinakis theorem

The Medvedev varieties M^+ and M^- cover the variety of Abelian ℓ-groups, so the Powell-Tsinakis theorem excludes a huge number of candidates for varieties with the AP.

In particular, there are uncountably many varieties of representable ℓ-groups that contain one of the Medvedev varieties.

There are, however, other representable covers of the variety of Abelian ℓ-groups. These are only partially classified.

Looking forward

It remains open whether there are any non-abelian varieties of ℓ-groups with the AP, and this has now been open for many decades.

It isn't clear what the right strategy for resolving this problem is.

One attack is to consider other covers of variety of abelian ℓ-groups.

Other attacks on the problem may focus on syntactic methods as in our proof of AP for abelian ℓ-groups.

Tomorrow

The Gurchenkov and Powell-Tsinakis theorems are essentially the state of the art on AP among ℓ-groups.

Tomorrow, we focus on applications of these results and zoom out into the wider context of cancellative residuated structures.

Thank you!