Amalgamation in lattice-ordered groups and cancellative residuated structures

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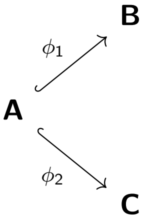
Amalgamation: the idea

This lecture series is about a very powerful algebraic property called amalgamation.

A class K of structures has the amalgamation property if every span $\langle \phi_1 \colon \mathbf{A} \to \mathbf{B}, \phi_2 \colon \mathbf{A} \to \mathbf{C} \rangle$ of structures in K can be completed in K:



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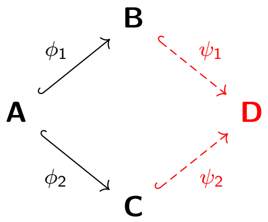

Amalgamation: motivation

Amalgamation is important in a huge number of different contexts:

- Fraïssé theory and its applications in (for example) dynamics and Ramsey theory
- Model completions and other areas of model theory
- The study of various kinds of syntactic interpolation properties

Amalgamation: history

The AP was first investigated for groups in [Schreier 1927] and, largely inaugurated by Jónsson's efforts, our understanding of AP in ordered algebras is especially deep:

- Exactly three varieties of lattices with AP [Day-Ježek 1984]
- Exactly eight varieties of Heyting algebras with the AP [Maksimova 1977]
- Lots of progress for other classes of residuated structures by many authors in the last decade, fueling some powerful general theory

Amalgamation: relation to CEP

Theorem (Kearnes 1989):

Let V be any congruence modular, residually small variety (i.e., there is a cardinal bound on the size of subdirectly irreducibles in V). If V has the AP, then V has the congruence extension property.

Amalgamation: sufficient conditions

Theorem (F.-Metcalfe 2024):

Let K be a subclass of a variety V satisfying

1. K is closed under isomorphisms and subalgebras;
2. every relatively subdirectly irreducible member of V belongs to K;
3. for any $\mathbf{B} \in \mathsf{V}$ and subalgebra $\mathbf{A}$ of $\mathbf{B}$, if $\Theta \in \mathrm{Con}\,\mathbf{A}$ and $\mathbf{A}/\Theta \in \mathsf{K}$, then there exists a $\Phi \in \mathrm{Con}\,\mathbf{B}$ such that $\Phi \cap A^2 = \Theta$ and $\mathbf{B}/\Phi \in \mathsf{K}$;
4. every span of finitely generated algebras in K has an amalgam in V.

Then V has the amalgamation property.

Amalgamation: characterization

Theorem (F.-Metcalfe 2024):

Let V be any quasivariety with the V-congruence extension property such that $\mathsf{V}_{\mathrm{FSI}}$ is closed under subalgebras. The following are equivalent:

1. V has the amalgamation property.
2. V has the one-sided amalgamation property.
3. $\mathsf{V}_{\mathrm{FSI}}$ has the one-sided amalgamation property.
4. Every span in $\mathsf{V}_{\mathrm{FSI}}$ has an amalgam in $\mathsf{V}_{\mathrm{FSI}} \times \mathsf{V}_{\mathrm{FSI}}$.
5. Every span of finitely generated algebras in $\mathsf{V}_{\mathrm{FSI}}$ has an amalgam in V.

Equational consequence

For a set of variables Y, we denote by $\mathbf{Tm}(Y)$ the term algebra over Y (reading the signature as given). For a variety V, we denote $\mathbf{F}_{\mathsf{V}}(Z)$ the free algebra in V generated by the set Z. If $\epsilon \in \mathbf{Tm}(Z)$, then $\bar{\epsilon}$ is the image of ϵ under the natural projection $\mathbf{Tm}(Z) \to \mathbf{F}_{\mathsf{V}}(Z)$.

Write $\mathrm{Eq}(Y)$ for the collection of equations in the variables Y. For $\Sigma \cup \{\epsilon\} \subseteq \mathrm{Eq}(Y)$ and K any class of algebras, define:

$$\Sigma \models^{Y}_{\mathsf{K}} \epsilon \iff \text{For each } \mathbf{A} \in \mathsf{K} \text{ and each homomorphism } h\colon \mathbf{Tm}(Y) \to \mathbf{A},\ \text{if } \Sigma \subseteq \ker(h) \text{ then } \epsilon \in \ker(h).$$

Equational consequence

Proposition (Metcalfe-Montagna-Tsinakis 2014):

Let V be a variety, $Y \subseteq Z$, and $\Sigma \cup \{\epsilon\} \subseteq \mathrm{Eq}(Y)$. Let Θ_{V}^{Z} be the kernel of the projection map $\mathbf{Tm}(Z) \to \mathbf{F}(Z)$. The following are equivalent:

1. $\Sigma \models_{\mathsf{K}}^{Z} \epsilon$;
2. $\Sigma \models_{\mathsf{K}}^{Y} \epsilon$;
3. $\epsilon \in \mathrm{Cg}^{\mathbf{Tm}(Z)}(\Sigma) \vee \Theta_{\mathsf{V}}^{Z}$;
4. $\bar{\epsilon} \in \mathrm{Cg}^{\mathbf{F}(Z)}(\bar{\Sigma})$;
5. $\epsilon \in \mathrm{Cg}^{\mathbf{Tm}(Y)}(\Sigma) \vee \Theta_{\mathsf{V}}^{Y}$;
6. $\bar{\epsilon} \in \mathrm{Cg}^{\mathbf{F}(Y)}(\bar{\Sigma})$.

Equational consequence

If $Y \subseteq Z$, then congruences on $\mathbf{F}(Y)$ extend to $\mathbf{F}(Z)$. So, the usual equational consequence can be defined by

$$\Sigma \models_{\mathsf{V}} \epsilon \iff \Sigma \models_{\mathsf{V}}^{Y} \epsilon \text{ for any } Y \supseteq \mathrm{Var}(\Sigma \cup \{\epsilon\}).$$

Assume that V is a variety with at least one constant symbol. We say that V has the equational deductive interpolation property (or EqDIP) if for any set of variables Y, whenever

$$\Sigma \cup \{\epsilon\} \subseteq \mathrm{Eq}(Y) \text{ and } \Sigma \models_{\mathsf{V}} \epsilon$$

then there exists $\Delta \subseteq \mathrm{Eq}(Y)$ such that

$$\Sigma \models_{\mathsf{V}} \Delta, \;\; \Delta \models_{\mathsf{V}} \epsilon, \text{ and } \mathrm{Var}(\Delta) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\epsilon).$$

Amalgamation and deductive interpolation

Theorem:

Let V be any variety with a constant in the language.

1. If V has AP, then V has EqDIP.
2. if V has the CEP and EqDIP, then V has AP.

Aside: the challenge of EqDIP

EqDIP is in many respects much more subtle than AP.

- There are many varieties with EqDIP that lack AP (e.g. semigroups).
- Establishing EqDIP is often extremely difficult by purely algebraic methods.
- Often, proofs of EqDIP factor through some nice syntactic presentation of the varieties in question (e.g. via proof theory).
- There is a notion of weak CEP under which EqDIP + WCEP holds iff AP holds.

Switching gears: residuated structures

Now we will narrow our perspective, focusing on some concrete varieties that tells us a lot about amalgamation and syntactic interpolation properties.

Residuated Lattices: the basics

A residuated lattice is an algebraic structure of the form $(A, \wedge, \vee, \cdot, \backslash, /, e)$ where

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, e)$ is a monoid, and
- for all $x, y, z \in A$,

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z/y.$$

We use all the expected terminology: Commutative, idempotent, totally ordered, linear, etc.

Semilinear: Subdirect product of totally ordered residuated lattices.

Knotted rules: $x \leq e$ (integral), $x \leq x^2$) (square inc), $x^2 \leq x$ (square decreasing).

Residuated Lattices: examples

Lots of familiar examples of residuated lattices, sometimes including expansions by a bottom element or an involution $\neg$.

- Boolean algebras and Heyting algebras, where product $\cdot$ is $\wedge$.
- Algebras of ideals of rings, where product is the product of ideals and division is the usual ideal division.
- Relation algebras in Tarski's sense.
- Various more exotic algebras associated to non-classical logics: MV-algebras, BL-algebras, Sugihara monoids, De Morgan monoids...

Residuated Lattices: properties

Residuated lattices are arithmetical: both congruence distributive and congruence permutable.

Nice characterization of congruence: correspond to convex normal subalgebras.

Commutative RLs also have the CEP, but generally non-commutative ones lack CEP.

In commutative RLs, $x \backslash y = y / x$ so usually written as $x \to y$.

Cancellativity

A residuated lattice is called cancellative if both of

$$xz = yz \Rightarrow x = y, \text{ and}$$

$$zx = zy \Rightarrow x = y$$

hold. Remarkably, in residuated lattices, cancellativity is equivalent to the identities

$$(xy)/y = x = y\backslash(yx),$$

so cancellative RLs form a variety. Particular examples are given by lattice-ordered groups.

Lattice-ordered groups

Lattice-ordered groups are a very old topic. Most often, an ℓ-group is defined as a group **G** with a lattice order $\leq$ such that

$$x \leq y \Rightarrow xz \leq yz,$$

$$x \leq y \Rightarrow zx \leq zy.$$

Equivalently, ℓ-groups can be realized as residuated lattices that satisfy the single equation

$$(e\backslash x)x = e,$$

in which case we have that $x\backslash e = e/x = x^{-1}$.

Lattice-ordered groups: the commutative case

Abelian ℓ-groups are quite well understood. They are generated as a variety by the ℓ-group of integers [Weinberg 1965].

Among the consequences of this: Every non-trivial variety of ℓ-groups contains the variety of abelian ℓ-groups.

Of course, abelian ℓ-groups also form a variety of cancellative semilinear residuated lattices.

In fact, when it comes to amalgamation, semilinear ℓ-groups occupy a rather special place among all varieties of residuated lattices.

The role of ℓ-groups

As we shall see:

- Abelian ℓ-groups have the AP.
- The entire variety of ℓ-groups does not have the AP.
- Abelian ℓ-groups are semilinear but general ℓ-groups are not.
- Gurchenkov 1997: Every variety of ℓ-groups with the AP is semilinear.
- But the variety of semilinear ℓ-groups doesn't have the AP [Glass-Saracino-Wood 1984] and there are uncountably many varieties of semilinear ℓ-groups without the AP [Powell-Tsinakis 1989].
- Almost everything known about AP in cancellative RLs comes from reducing to the ℓ-group case.

What we know for semilinear RLs

Here * means that the same is true for bounded and involutive expansions

Variety V	Abbreviation	AP	$\lvert\Omega(V)\rvert$
Semilinear residuated lattices	SRL	no*	$2^{\aleph_0}$
Commutative SRL	CSRL	no*	$\geq \aleph_0$
Idempotent SRL	1SRL	no	$2^{\aleph_0}$
Idempotent CSRL	1CSRL	yes	60
Gödel algebras	GA	yes	4
Relative Stone algebras	RSA	yes	3
Sugihara monoids	SM	yes	9
Odd Sugihara monoids	OSM	yes	3
$\langle m, n\rangle$-knotted SRL ($m \geq 1$, $n \geq 0$)		no*	$2^{\aleph_0}$
$\langle m, n\rangle$-knotted CSRL ($m \geq 1$, $n \geq 0$)		no*	?
n-potent SRL ($n \geq 2$)		no*	$2^{\aleph_0}$
n-potent CSRL ($n \geq 2$)		no*	≥ 60
MTL-algebras	MTL	no	$\geq \aleph_0$
De Morgan monoids	DMM	no	$\geq \aleph_0$
Semilinear DMM	SDMM	no	$\geq \aleph_0$
Cancellative SRL	CanSRL	no	≥ 3
Commutative CanSRL	CanCSRL	?	≥ 3
Integral CanSRL		?	≥ 2
Lattice-ordered groups	LG	no	≥ 2
Abelian LG	ALG	yes	2
Representable LG	RLG	no	≥ 2
MV-algebras	MV	yes	$\aleph_0$
Wajsberg hoops	WH	yes	$\aleph_0$
BL-algebras	BL	yes	$\aleph_0$
Basic hoops	BH	yes	$\aleph_0$

W. Fussner and S. Santschi, Amalgamation in Semilinear Residuated Lattices, `https://arxiv.org/abs/2407.21613`, 2024.

Amalgamation in ℓ-groups: the commutative case

There are lots of proofs that abelian ℓ-groups have the AP.

- Pierce 1973, with the Hahn embedding theorem much later.
- Algebraic proofs due to Powell and Tsinakis 1983, 1989.
- Using quantifier elimination [Weispfenning 1989].
- Metcalfe-Montagna-Tsinakis 2023 using EqDIP, which we give.

Amalgamation in ℓ-groups: the commutative case

Theorem (see MMT 2023):
The variety of abelian ℓ-groups has the AP.

Proof: Abelian ℓ-groups are a variety of RLs with the CEP, so it is enough to show that they have the equational deductive interpolation property.

Proof

For this, it is enough to show that for any term α and any $x \in \mathrm{Var}(\alpha)$, there exists a term γ such that $\mathrm{Var}(\gamma) \subseteq \mathrm{Var}(\alpha) - \{x\}$ and for all $\beta \in \mathbf{Tm}$ with $x \notin \mathrm{Var}(\beta)$, we have

$$e \leq \alpha \models e \leq \beta \iff e \leq \gamma \models e \leq \beta.$$

This is because:

1. Every abelian ℓ-group equation can be written in the form $e \leq \epsilon$, and
2. The equational consequence relation of abelian ℓ-groups is finitary.

Proof

Note that abelian ℓ-groups satisfy the distributivity laws

$$x + (y \wedge z) = (x + y) \wedge (x + z),$$

$$x + (y \vee z) = (x + y) \vee (x + z).$$

Further, all ℓ-groups are distributive as lattices. Using these distributivity properties, we may assume without loss of generality that for some $n \geq 1$ and terms $\alpha', \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_k$ not containing x,

$$\alpha = \alpha' \wedge \bigwedge_{i=1}^{m} (\alpha_i + nx) \wedge \bigwedge_{j=1}^{k} (\beta_j - nx).$$

The correct definition of γ is then found by setting

$$\gamma = \alpha' \wedge \bigwedge \{\alpha_i + \beta_j \mid 1 \leq i \leq m, 1 \leq j \leq k\}.$$

Thank you!