Representing Sugihara monoids with binary relations

Andrew Craig Wilmari Morton Claudette Robinson

UNIVERSITY OF JOHANNESBURG

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Co-authors & papers

- A. Craig, W. Morton, C. Robinson, *Representability for distributive quasi relation algebras via generalised ordinal sums*, submitted. `https://arxiv.org/abs/2503.06657`
- A. Craig, C. Robinson, *Representing Sugihara monoids via weakening relations*, Fundamenta Informaticae, to appear. `https://arxiv.org/abs/2310.12935`

See also:

- A. Craig, C. Robinson, *Representable distributive quasi relation algebras*, Algebra Universalis 86:12 (2025)

Outline

1. Residuated lattices and DInFL-algebras
2. Generalised ordinal sums: $\mathbf{K}[\mathbf{L}]$
3. Representing DInFL-algebras with binary relations
4. Application: representing Sugihara monoids

Residuated lattices and DInFL-algebras

$\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ is a residuated lattice if $\langle A, \wedge, \vee \rangle$ is a lattice and $\langle A, \cdot, 1 \rangle$ is a monoid such that:

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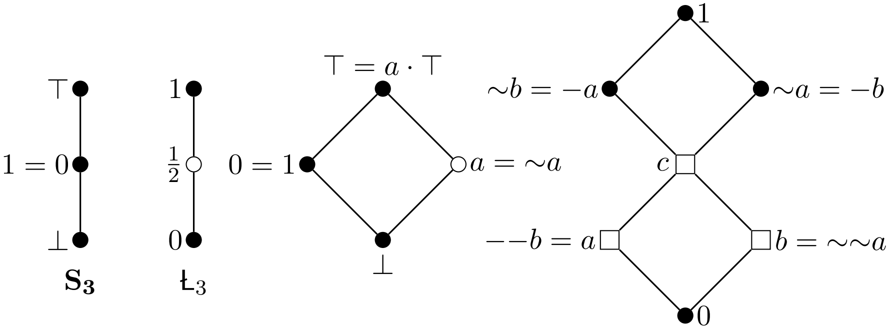
Goal: find relational representations of DInFL-algebras.

Examples of DInFL-algebras



Idempotent elmts = solid nodes, non-idempotents = empty nodes.
Circles = central elements, squares=non-central.

See full list up to cardinality 8 by C., Jipsen, Robinson: DInFL1.pdf

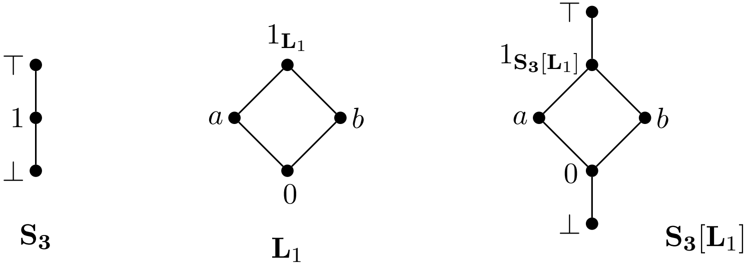
Others examples include: MV-algebras, relation algebras, Sugihara monoids.

A three-element chain

Descending Jahňací štít – SSAOS 2022

Generalised ordinal sums (cf. Galatos 2004)

Example 1:



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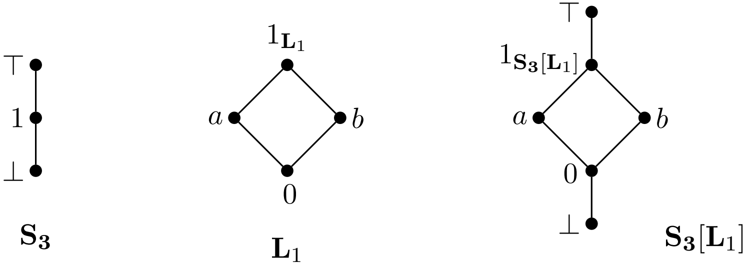
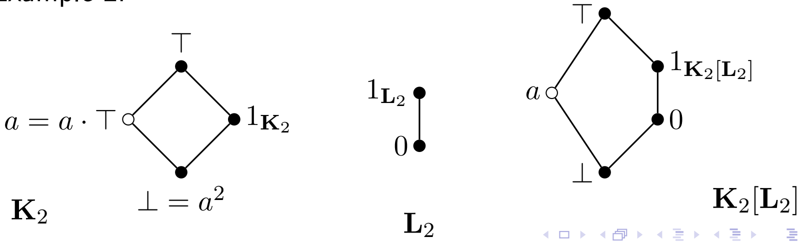


Example 2:



Generalised ordinal sums

Let $\mathbf{K}$ be a DInFL-algebra.
$1_{\mathbf{K}}$ is totally irreducible if for all non-nullary operations f, $f(a_1, \ldots, a_n) = 1_{\mathbf{K}}$ implies $a_i = 1_{\mathbf{K}}$ for some $i \in \{1, \ldots, n\}$.

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A residuated lattice $\mathbf{K}$ is conic if $a \leqslant 1$ or $1 \leqslant a$ for all $a \in K$.

Theorem

Let $\mathbf{K}$ and $\mathbf{L}$ be DInFL-algebras such that $\mathbf{K}$ is conic and $1_{\mathbf{K}}$ is totally irreducible, then their generalised ordinal sum $\mathbf{K}[\mathbf{L}]$ is a DInFL-algebra.

Sugihara monoids

A Sugihara monoid is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, 1, \sim \rangle$ such that $\langle A, \wedge, \vee, \cdot, \to, 1 \rangle$ is a commutative distributive idempotent residuated lattice, and for all $a, b \in A$:

- $\sim\sim a = a$
- $a \to \sim b = b \to \sim a$

Sugihara monoids provide algebraic semantics for $\mathbf{RM^t}$ ($\mathbf{R}$-mingle with added Ackermann constant)

Can be considered as commutative idempotent DInFL-algebras.

Finite Sugihara chains

If $n = 2k$ for $k > 0$ then $S_n = \{a_{-k}, \ldots, a_{-1}, a_1, \ldots, a_k\}$
If $n = 2k + 1$ for $k > 0$ then $S_n = \{a_{-k}, \ldots, a_{-1}, a_0, a_1, \ldots a_k\}$

- $a_i \wedge a_j = a_{\mathsf{min}\{i,j\}}$ and $a_i \vee a_j = a_{\mathsf{max}\{i,j\}}$.
- $\sim a_j = a_{-j}$

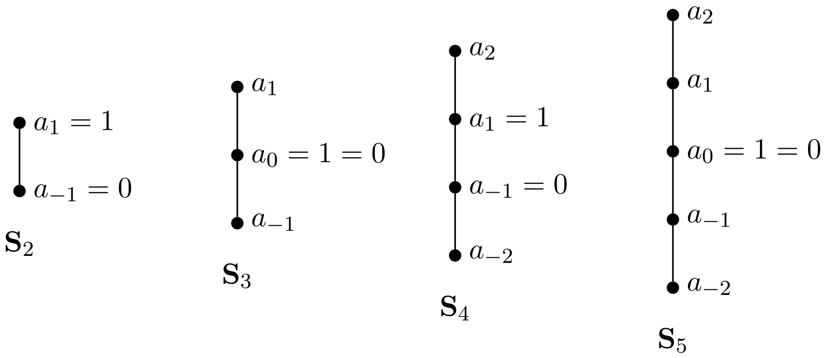
$$a_i \cdot a_j = \begin{cases} a_i & \text{if } |j| < |i| \\ a_j & \text{if } |i| < |j| \\ a_{\mathsf{min}\{i,j\}} & \text{if } |j| = |i|. \end{cases}$$

$$a_i \to a_j = \begin{cases} \sim a_i \vee a_j & \text{if } i \leqslant j \\ \sim a_i \wedge a_j & \text{if } i > j. \end{cases}$$

If n odd, then $1 = a_0$.
If n even, then $1 = a_1$.

$$\mathbf{S}_n = \langle S_n, \wedge, \vee, \cdot, \to, 1, \sim \rangle$$

Finite Sugihara chains via K[L]



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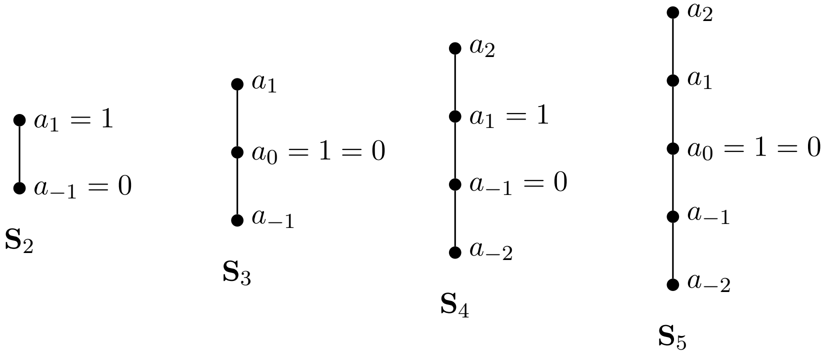

Proposition

Let $\mathbf{K} = \mathbf{S}_n$ *for* n *odd, and* $\mathbf{L} = \mathbf{S}_m$ *for* $m \geqslant 2$*, then* $\mathbf{K}[\mathbf{L}] \cong \mathbf{S}_{n+m-1}$.

Representable DInFL-algebras

Let $(X, \leqslant)$ be a poset and $\leqslant \subseteq E$ an equivalence relation on X.
For $(x, y), (w, z) \in E$ define:

$$(x, y) \preccurlyeq (w, z) \quad \iff \quad w \leqslant x \ \& \ y \leqslant z$$

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Then $\mathbf{E} = (E, \preccurlyeq)$ is a poset and $\langle \mathsf{Up}(\mathbf{E}), \cap, \cup, ;, \leqslant \rangle$ is a distributive residuated lattice.

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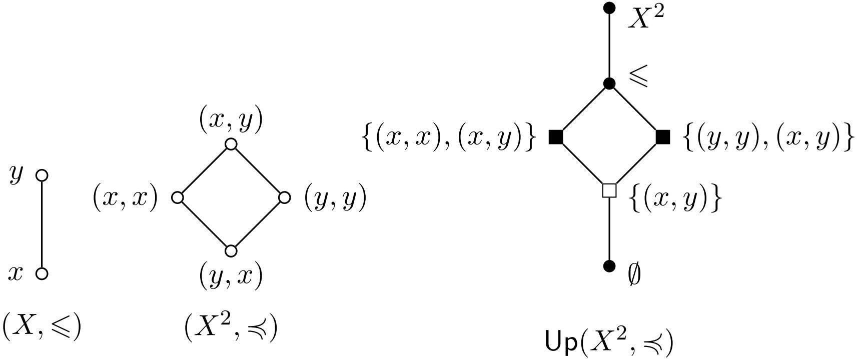

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Theorem (C., Robinson 2025, Theorem 3.15)

Let $\mathbf{X} = (X, \leqslant)$ *be a poset and* E *an equivalence relation on* X *such that* $\leqslant \subseteq E$. *Let* $\alpha : X \to X$ *be an order automorphism of* $\mathbf{X}$ *s.t.* $\alpha \subseteq E$. *Set* $1 = \leqslant$ *and* $0 = \alpha \,;\, \leqslant^{c\smile}$. *For* $R \in \mathsf{Up}(\mathbf{E})$, *define* ${\sim}R = R^{c\smile} \,;\, \alpha$, $-R = \alpha \,;\, R^{c\smile}$. *Then*

- $\mathfrak{D}(\mathbf{E}) = \langle \mathsf{Up}\,(\mathbf{E})\,, \cap, \cup, ;, \sim, -, 1, 0 \rangle$ *is a DInFL-algebra;*
- $\mathfrak{D}(\mathbf{E})$ *is cyclic iff* α *is the identity map.*

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Algebras of the form $\mathfrak{D}(\mathbf{E})$ are equivalence DInFL-algebras.
If $E = X^2$ it is a full DInFL-algebra.
Classes denoted EDInFL and FDInFL.

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Definition (C., Robinson 2025)

A DInFL-algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \sim, -, 1, 0 \rangle$ is *representable* if $\mathbf{A} \in \mathbb{ISP}\,(\mathrm{FDInFL})$ or, equivalently, $\mathbf{A} \in \mathbb{IS}\,(\mathrm{EDInFL})$.

Examples

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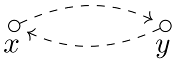
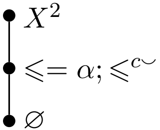


$\mathbf{E} = (X^2, \preccurlyeq)$

$\mathbf{S}_3 \hookrightarrow \mathfrak{D}(\mathbf{E})$



Generalised ordinal sums and representability

Theorem

Let $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \sim, -, 1, 0\rangle$ *be a representable DInFL-algebra. Then* $\mathbf{S}_3\left[\mathbf{L}\right]$ *is a representable DInFL-algebra.*

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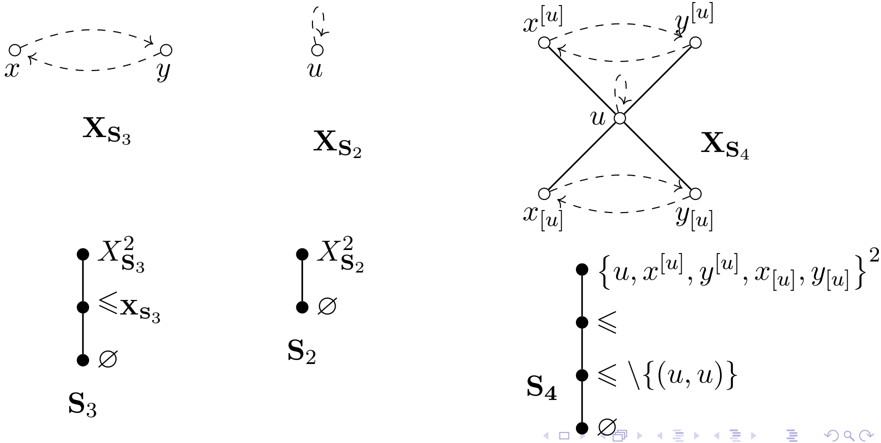

Representing finite Sugihara chains

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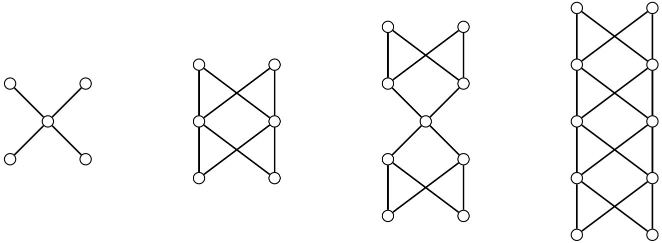
Corollary

All finite Sugihara chains are representable. Moreover, the posets used in the representations are finite.



Posets used to represent $\mathbf{S_4}$ to $\mathbf{S_7}$ ($E = X^2$, α swaps left-to-right)

Ultraproducts of representable DInFL-algebras

Let $\{ (X_i, \leqslant_i, E_i, \alpha_i) \mid i \in I\}$ be a set of posets with equivalence relations E_i and order automorphisms α_i s.t. $\leqslant_i \subseteq E_i$ and $\alpha_i \subseteq E_i$.

For $\mathcal{F}$ an ultrafilter on I, form an ultraproduct of the $\{(X_i, \leqslant_i, E_i, \alpha_i) \mid i \in I\}$:

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$\mathbb{P}_{\mathbb{U}}(\mathsf{RDInFL}) = \mathsf{RDInFL}$

Proof.

Consider $\{\, \mathbf{A}_i \mid i \in I \,\}$ and $\mathcal{F}$ an ultrafilter on I. Each $\mathbf{A}_i \hookrightarrow \mathfrak{D}(X_i, \leqslant_i, E_i, \alpha_i)$. Embed $\prod\{\mathbf{A}_i \mid i \in I\}/\theta_{\mathcal{F}}$ into $\mathfrak{D}(Y, \leqslant_Y, E_Y, \alpha_Y)$. □

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Key facts:

Fin. gen. subalg. of $\mathbf{S} = (\mathbb{Z}, \wedge, \vee, \cdot, \rightarrow, 1, \sim)$ are $\mathbf{S}_{2n+1}$, $n \in \omega$

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Let $I = \{2k+1 \mid k \in \omega\}$ and $\mathbf{A} = \prod\{\, \mathbf{S}_i \mid i \in I \,\}$. Then $\exists\, \mathcal{F}$ s.t. $\mathbf{S} \hookrightarrow \mathbf{A}/\theta_{\mathcal{F}}$.

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Hence, SM $= \mathbb{ISP}(\mathbf{S}, \mathbf{S}^*) \subseteq \mathbb{ISP}(\mathsf{RDInFL}) = \mathsf{RDInFL}$. □

References

1. A .Craig, P. Jipsen, C. Robinson: Distributive involutive residuated lattices up to cardinality 8. Online: DInFL1.pdf
2. A. Craig, C. Robinson: Representable distributive quasi relation algebras, *Algebra Universalis*, 86:12 (2025)
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