Multiplicatively Transtable Conic Sections with respect to Fixed Coefficients

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Contents

INTRODUCTION

WHAT IS TRANSTABILITY?

► **the basic method of preservation**

ADDITIVE TRANSTABILITY:

⊕ $F(x+k, y-k) = F(x,y)$ for $k \in \mathbb{R}$ and $x, y \in \mathbb{R}$.

MULTIPLICATIVE TRANSTABILITY:

⊙ $F\left(x \cdot k, \frac{y}{k}\right) = F(x,y)$ for $k \in \mathbb{R} \setminus \{0\}$ and $x, y \in \mathbb{R}$.

SHIFT TRANSTABILITY:

$\uparrow\downarrow$ $F(w,z) = F(x,y)$ for $x \prec w$, $z \prec y$ and $w, x, y, z \in L$.

For example: Polynomials

Definition

We say that the polynomial $p(x)$ is *Transtable* with the polynomial $r(x)$ if the polynomial $r(x)$ was created from the polynomial $p(x)$ by **multiplying** the (nonzero) constant $k \in \mathbb{R}$ to one coefficient, and by **dividing** the same constant k from another coefficient.

$\Rightarrow$ **The product of the coefficients is preserved (except for zero coefficients).**

Remark:

- Both polynomials must be of **the same degree**.
- Polynomials with **zero coefficients** are a problem because the product of the coefficients is equal to zero and the polynomials **may not be transtable**.
- This problem can be solved by dividing the entire set of polynomials into smaller classes based on **the number and position of the zero coefficients**.

TRANSTABILITY FOR CONIC SECTIONS

Complete Transtability

Definition

Two conic sections K and L are said to be ***Multiplicatively Transtable*** if there exists a constant $k \in \mathbb{R} \setminus \{0\}$ such that by **multiplying one coefficient** and **dividing another coefficient** of the conic section K by the constant k we get the conic section L.

Specifically, for conic sections K_0 and L_0 with non-zero coefficients **the product of their coefficients is the same**, i.e. for conic sections

$$K_0: \quad a_1x^2 + b_1y^2 + c_1xy + d_1x + e_1y + f_1 = 0,$$
$$L_0: \quad a_2x^2 + b_2y^2 + c_2xy + d_2x + e_2y + f_2 = 0,$$

this condition holds: $a_1 \cdot b_1 \cdot c_1 \cdot d_1 \cdot e_1 \cdot f_1 = a_2 \cdot b_2 \cdot c_2 \cdot d_2 \cdot e_2 \cdot f_2$.

Complete Transtability

Remark:

- It is evident that the entire set of multiplicative transtable conic sections **cannot be divided into equivalence classes** based on the product of coefficients, **with zero coefficients being the primary issue**. However ...
- The set of transtable conic sections can be denoted $[k]$ by a real number k representing the **product of non-zero coefficients**.
- For a conic section with a coefficient product equal to 0, **additional information about the position of the zero coefficients is required**, for instance $\left[0_{y^2,x,const}\right]$.
- Each non-zero real number corresponds to a single class, but the zero class $[0]$ must be divided into 63 subclasses – these are **equivavalence classes** for entire set of transtable conic sections.

Complete Transtability

For Example

1. $K : \mathbf{1x^2 + 1y^2 - 1 = 0},\ K \in [0_{xy,x,y}]$

$$K_2 :\ 2x^2 + \frac{1}{2}y^2 - 1 = 0$$

$$K_3 :\ \frac{1}{2}x^2 + 2y^2 - 1 = 0$$

$$K_4 :\ 2x^2 + 1y^2 - \frac{1}{2} = 0$$

$$K_5 :\ \frac{1}{2}x^2 + 1y^2 - 2 = 0$$

$$K_6 :\ 1x^2 + 2y^2 - \frac{1}{2} = 0$$

$$K_7 :\ 1x^2 + \frac{1}{2}y^2 - 2 = 0$$

$$K_8 :\ -2x^2 + 1y^2 + \frac{1}{2} = 0$$

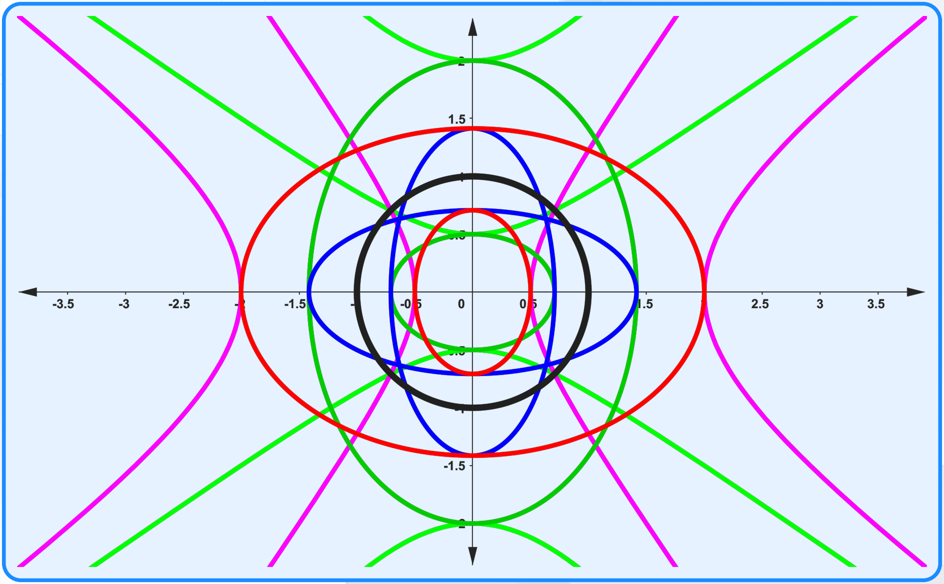
$$K_9 :\ -\frac{1}{2}x^2 + 1y^2 + 2 = 0$$

$$K_{10} :\ 1x^2 - 2y^2 + \frac{1}{2} = 0$$

$$K_{11} :\ 1x^2 - \frac{1}{2}y^2 + 2 = 0$$

- Shift of the two different **NON-ZERO** coefficients by 2.

Complete Transtability



Complete Transtability

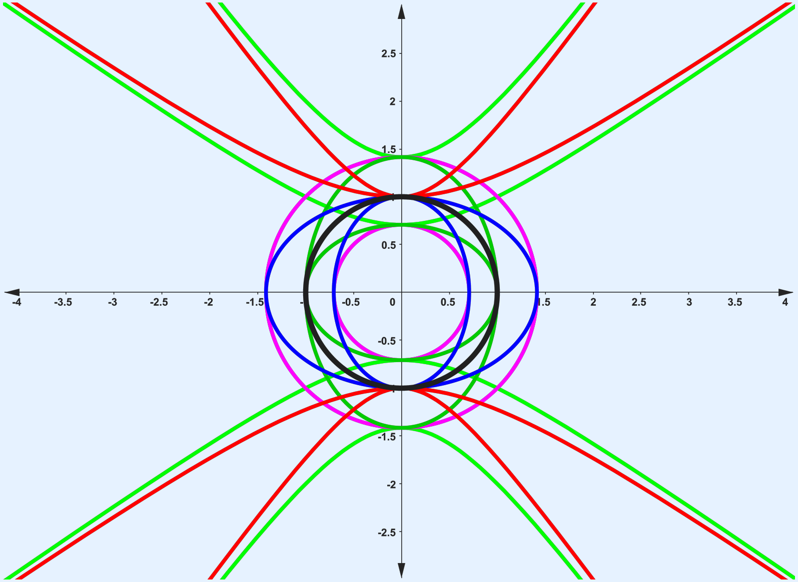
For Example

2. $K : \mathbf{1x^2 + 1y^2 - 1 = 0},\ K \in [0_{xy,x,y}]$

$$K_{12} :\ 2x^2 + 1y^2 - 1 = 0$$
$$K_{13} :\ \frac{1}{2}x^2 + 1y^2 - 1 = 0$$
$$K_{14} :\ -2x^2 + 1y^2 - 1 = 0$$
$$K_{15} :\ -\frac{1}{2}x^2 + 1y^2 - 1 = 0$$
$$K_{16} :\ 1x^2 + 2y^2 - 1 = 0$$
$$K_{17} :\ 1x^2 + \frac{1}{2}y^2 - 1 = 0$$
$$K_{18} :\ 1x^2 - 2y^2 - 1 = 0$$
$$K_{19} :\ 1x^2 - \frac{1}{2}y^2 - 1 = 0$$
$$K_{20} :\ 1x^2 + 1y^2 - 2 = 0$$
$$K_{21} :\ 1x^2 + 1y^2 - \frac{1}{2} = 0$$

- Shift of the **ANY** two different coefficients by 2.

Complete Transtability



Partial Transtability

Definition

A conic section K in the form

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

is said to be ***Multiplicative Transtable with respect to x^2 and y^2*** with a conic section L if there **exists** $k \in \mathbb{R} \setminus \{0\}$ such that the conic section L has the form

$$(a \cdot k)\, x^2 + \frac{b}{k}\, y^2 + cxy + dx + ey + f = 0.$$

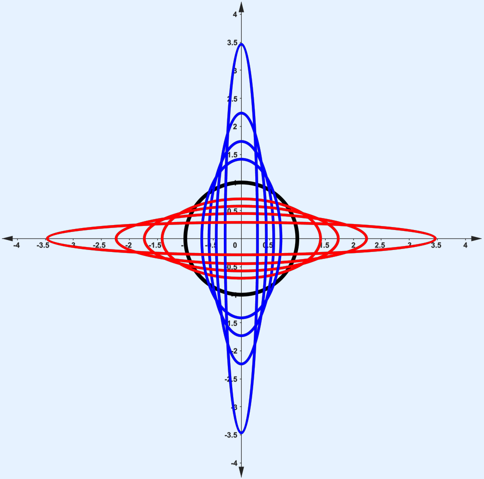
Remark:

- Definition induces another 14 partial transtabilities.
- Complete transtability implies any partial transtability, but the reverse is not valid.

Partial Transtability

For Example

3. $K : \mathbf{1x^2 + 1y^2 - 1 = 0},\ K \in [1]$ for x^2 and y^2



INTERSECTION OF PARTIALLY TRANSTABLE CONIC SECTIONS

Intersection – Partially Transtable Conic Sections

Theorem

Each two partially transtable conic sections have common points if and only if they are transtable with respect to x^2 and xy or y^2 and xy or x^2 and x or y^2 and y or xy and x or xy and y, i.e. ***there are no general common points for the 9 partial transtabilities****.*

Proof:

The proof is divided into **15** parts, for each partial transtability. Consider a conic section in the form:

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

Then ...

INTERSECTION – TRANSTABILITY WITH RESPECT TO x^2 AND y^2

Consider two arbitrary transtable conic sections K and L with respect to x^2 and y^2 :

$$K : ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L : (a \cdot k)\, x^2 + \frac{b}{k}\, y^2 + cxy + dx + ey + f = 0.$$

In general **there are no intersections**. However, if we restrict some coefficients, we get:

1. The case $a = 0$ implies $y = 0$:

$$P_1^{x^2,y^2} = \left[-\frac{f}{d}, 0\right]$$

2. The case $b = 0$ implies $x = 0$:

$$P_2^{x^2,y^2} = \left[0, -\frac{f}{e}\right]$$

Intersection – Transtability with respect to x^2 and y^2

For Example

1. $K : -4x^2 + 7y^2 - 4xy + 5x + 2y - 4 = 0$

$$K_2 : 12x^2 - \frac{7}{3}y^2 - 4xy + 5x + 2y - 4 = 0,$$

$$K_3 : -\frac{4}{3}x^2 + 21y^2 - 4xy + 5x + 2y - 4 = 0,$$

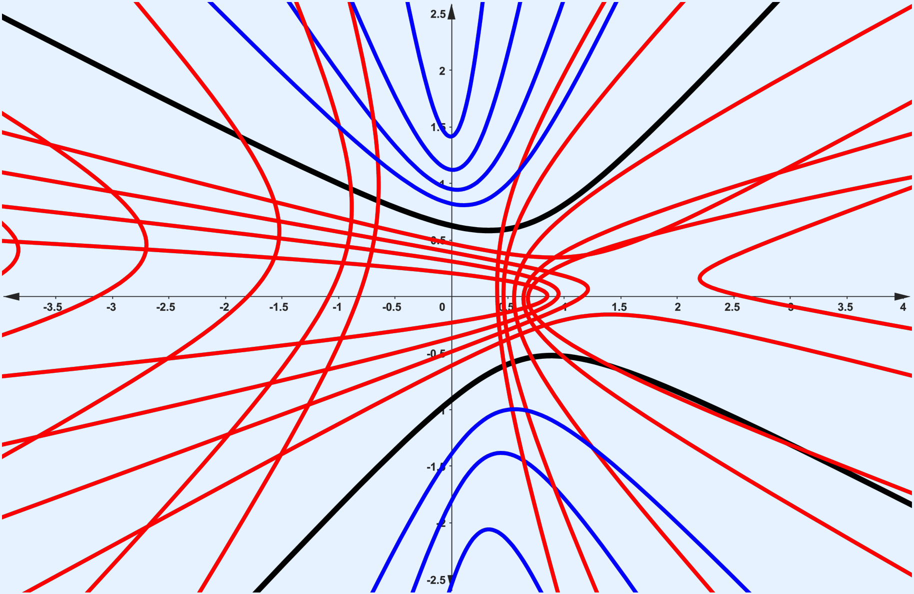
$$K_4 : -20x^2 + \frac{7}{5}y^2 - 4xy + 5x + 2y - 4 = 0,$$

$$K_5 : -\frac{4}{5}x^2 + 35y^2 - 4xy + 5x + 2y - 4 = 0,$$

$$K_6 : -\frac{1}{3}x^2 + 84y^2 - 4xy + 5x + 2y - 4 = 0,$$

The Common Points: None

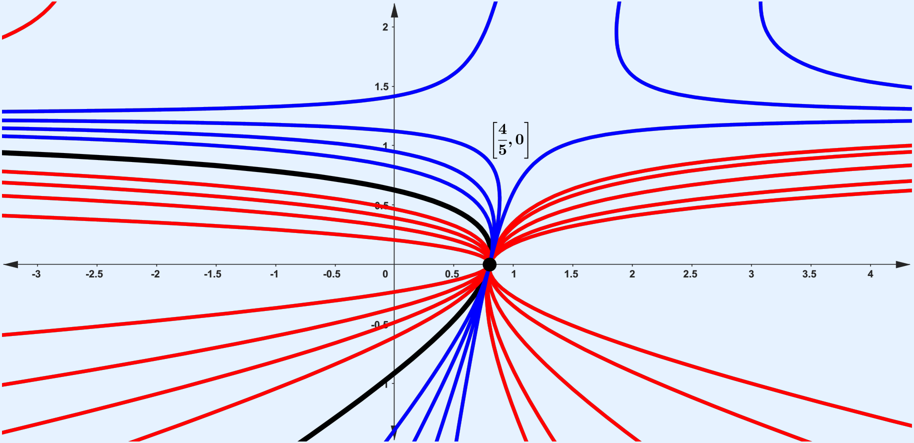
INTERSECTION – TRANSTABILITY WITH RESPECT TO x^2 AND y^2



INTERSECTION – TRANSTABILITY WITH RESPECT TO x^2 AND y^2

FOR EXAMPLE

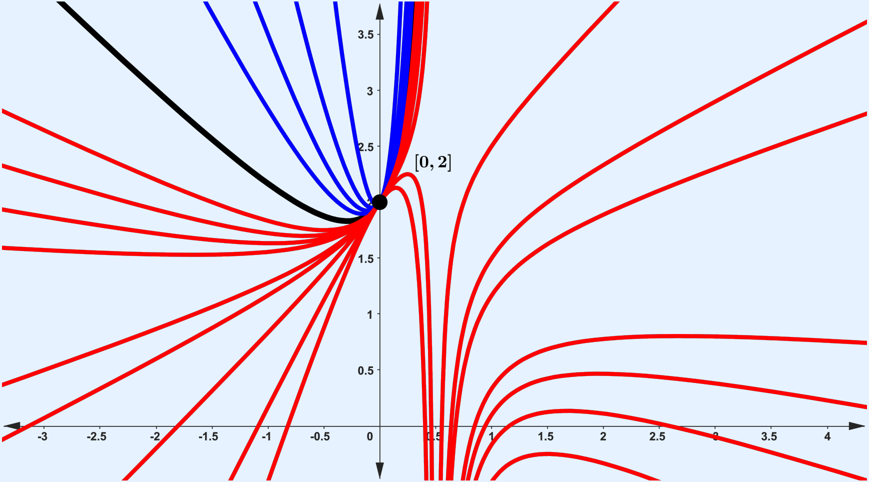
2. $L : 7y^2 - 4xy + 5x + 2y - 4 = 0$



INTERSECTION – TRANSTABILITY WITH RESPECT TO x^2 AND y^2

FOR EXAMPLE

3. $M : -4x^2 - 4xy + 5x + 2y - 4 = 0$



Intersection – Transtability with respect to x^2 and xy

Consider two arbitrary transtable conic sections K and L with respect to x^2 and xy :

$$K : ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L : (a \cdot k)\, x^2 + by^2 + \frac{c}{k} xy + dx + ey + f = 0.$$

Then we obtain the **two** intersection points:

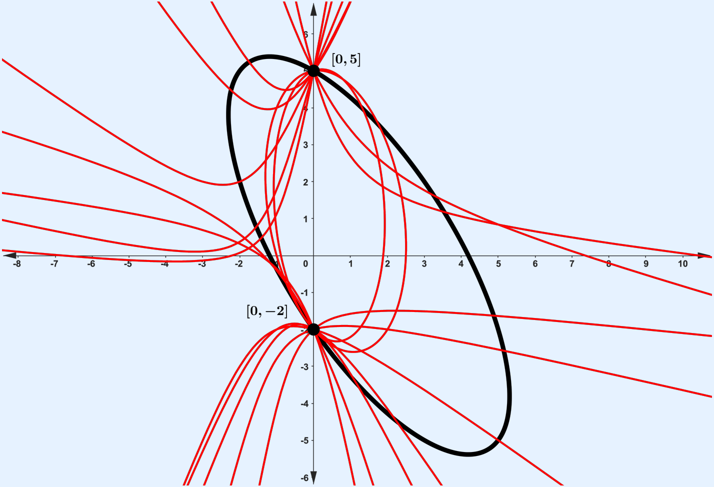
$$P_{1,2}^{x^2,xy} = [0, y]\,,$$

where y is the root of the quadratic equation: $by^2 + ey + f = 0$. For $a = 0$, we get a **third** intersection:

$$P_3^{x^2,xy} = \left[-\frac{f}{d}, 0\right].$$

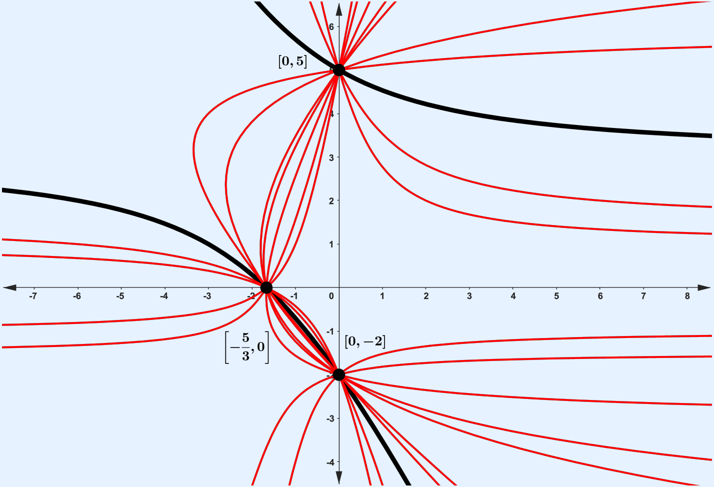
FOR EXAMPLE

4. $K: 2x^2 + y^2 + 2xy - 6x - 3y - 10 = 0$



FOR EXAMPLE

5. $L : y^2 + 2xy - 6x - 3y - 10 = 0$



INTERSECTION – TRANSTABILITY WITH RESPECT TO x^2 AND x

Consider two arbitrary transtable conic sections K and L with respect to x^2 and x :

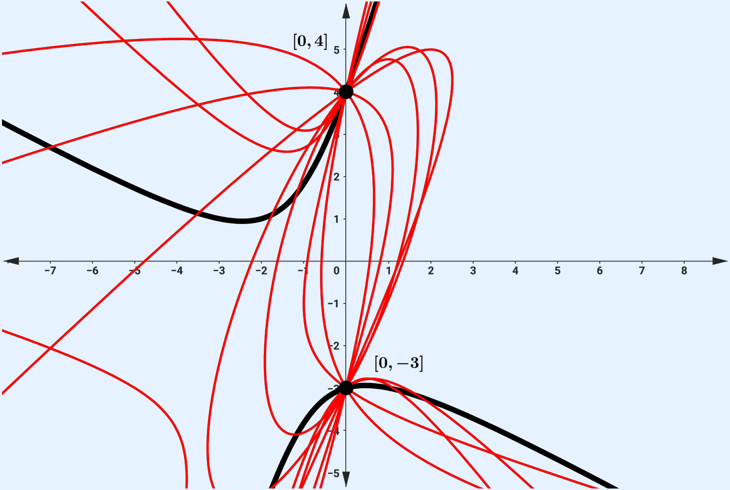
$$K : ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L : (a \cdot k)\, x^2 + by^2 + cxy + \frac{d}{k}\, x + ey + f = 0.$$

Then we obtain the **two** intersection points:

$$P_{1,2}^{x^2,x} = P_{1,2}^{x^2,xy}$$

FOR EXAMPLE

6. $K : -2x^2 + y^2 - 3xy - 7x - 1y - 12 = 0$



INTERSECTION – TRANSTABILITY WITH RESPECT TO x^2 AND y

Consider two arbitrary transtable conic sections K and L with respect to x^2 and y :

$$K : ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L : (a \cdot k)\, x^2 + by^2 + cxy + dx + \frac{e}{k}\, y + f = 0.$$

In general **there are no intersections**. However, if we restrict some coefficients, we get:

1. The case $a = 0$ implies $y = 0$:

$$P_1^{x^2,y} = \left[-\frac{f}{d}, 0\right]$$

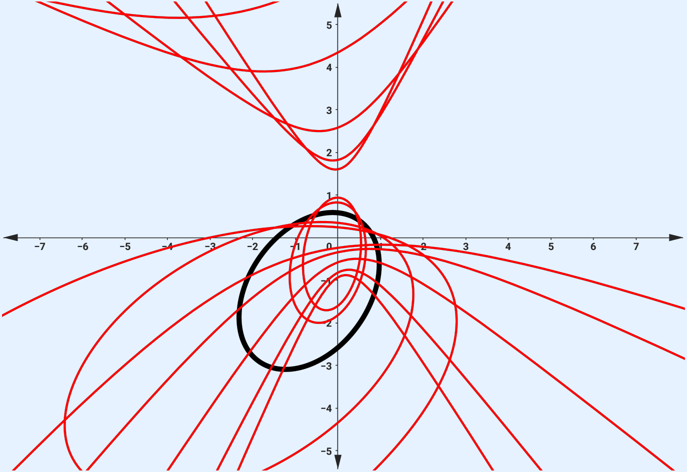
2. The case $e = 0$ implies $x = 0$:

$$P_{2,3}^{x^2,y} = [0, y]\,,$$

where y is the root of the quadratic equation: $by^2 + f = 0$.

FOR EXAMPLE

7. $K : 5x^2 + 4y^2 - 3xy + 3x + 8y - 6 = 0$



Intersection – Transtability with respect to x^2 and $const$

Consider two arbitrary transtable conic sections K and L with respect to x^2 and $const$:

$$K : ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L : (a \cdot k)\, x^2 + by^2 + cxy + dx + ey + \frac{f}{k} = 0.$$

In general **there are no intersections**. However, if we restrict some coefficients, we get:

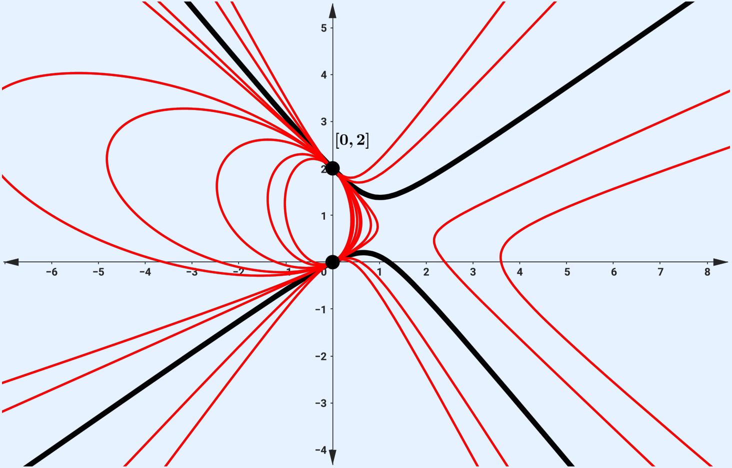
- The case $f = 0$ implies $x = 0$:

$$P_{1,2}^{x^2,const} = [0, y]$$

  where y is the root of the quadratic equation: $by^2 + ey = 0$.

FOR EXAMPLE

7. $K : -5x^2 + 6y^2 + 3xy + 6x - 12y = 0$



INTERSECTION – TRANSTABILITY WITH RESPECT TO xy AND x

Consider two arbitrary transtable conic sections K and L with respect to xy and x :

$$K : ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L : ax^2 + by^2 + (c \cdot k)\, xy + \frac{d}{k}\, x + ey + f = 0.$$

Then we obtain the **two** intersection points:

$$P_{1,2}^{xy,x} = P_{1,2}^{x^2,xy}$$

If we restrict $d = 0$, we get:

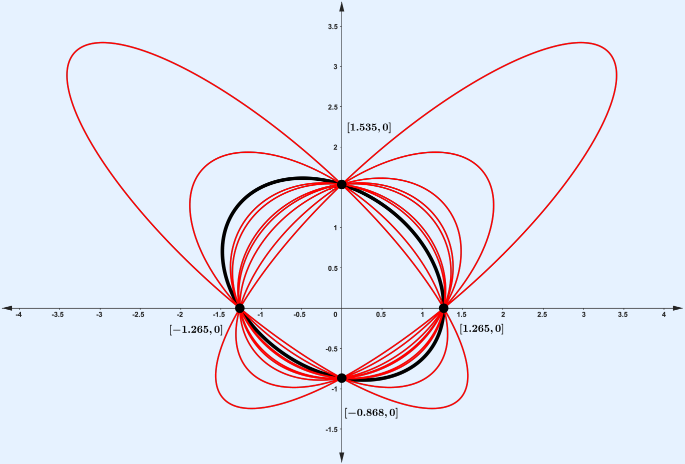
$$P_{3,4}^{xy,x} = P_{2,3}^{y^2,x} = [x, 0]$$

where x is the root of the quadratic equation: $ax^2 + f = 0$.

INTERSECTION – TRANSTABILITY WITH RESPECT TO xy AND x

FOR EXAMPLE

8. $K: 5x^2 + 6y^2 + 3xy - 4y - 8 = 0$



Intersection – Transtability with respect to xy and $const$

Consider two arbitrary transtable conic sections K and L with respect to xy and $const$:

$$K : ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L : ax^2 + by^2 + (c \cdot k)\, xy + dx + ey + \frac{f}{k} = 0.$$

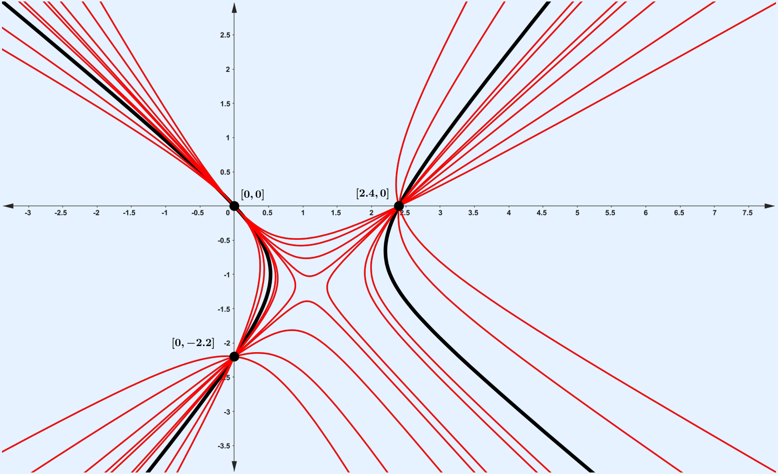
In general **there are no intersections**. However, if we restrict some coefficients, we get:

- The case $f = 0$ implies $x = 0$ or $y = 0$:

$$P_1^{xy,const} = \left[0, -\frac{e}{b}\right] \quad P_2^{xy,const} = \left[-\frac{d}{a}, 0\right] \quad P_3^{xy,const} = [0, 0]$$

FOR EXAMPLE

9. $K : 5x^2 - 5y^2 + 2xy - 12x - 11y = 0$



INTERSECTION – TRANSTABILITY WITH RESPECT TO x AND y

Consider two arbitrary transtable conic sections K and L with respect to x and y :

$$K : ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L : ax^2 + by^2 + cxy + (d \cdot k)\, x + \frac{e}{k}\, y + f = 0.$$

In general **there are no intersections**. However, if we restrict some coefficients, we get:

1. The case $d = 0$ implies:

$$P_{1,2}^{x,y} = [x, 0]\,,$$

   where x is the root of the quadratic equation: $ax^2 + f = 0$.
2. The case $e = 0$ implies:

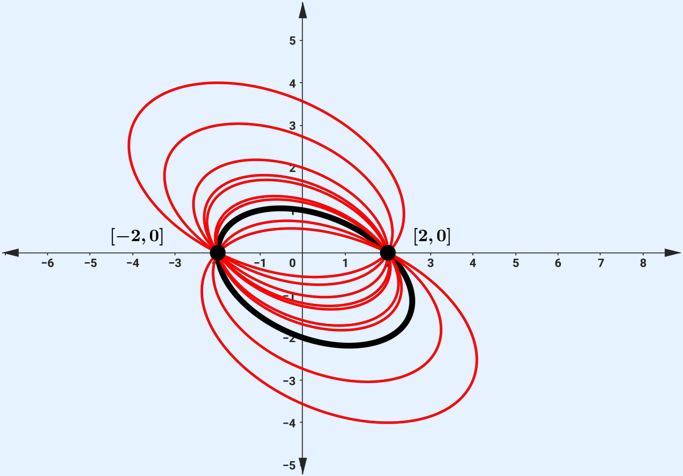
$$P_{3,4}^{x,y} = [0, y]\,,$$

   where y is the root of the quadratic equation: $by^2 + f = 0$.

INTERSECTION – TRANSTABILITY WITH RESPECT TO x AND y

FOR EXAMPLE

10. $K : 2x^2 + 4y^2 + 2xy + 4y - 8 = 0$



INTERSECTION – TRANSTABILITY WITH RESPECT TO x AND $const$

Consider two arbitrary transtable conic sections K and L with respect to x and $const$:

$$K : ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L : ax^2 + by^2 + cxy + (d \cdot k)\, x + ey + \frac{f}{k} = 0.$$

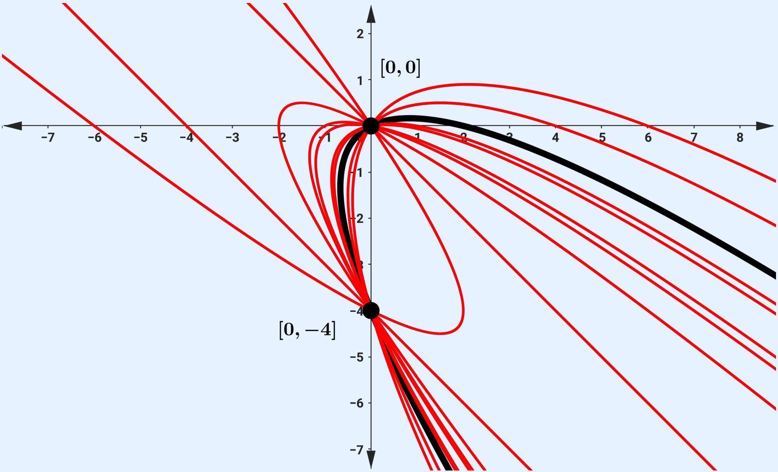
In general **there are no intersections**. However, if we restrict some coefficients, we get:

- The case $f = 0$ implies $x = 0$:

$$P_1^{x,const} = \left[0, -\frac{e}{b}\right] \quad P_2^{x,const} = [0, 0]$$

FOR EXAMPLE

11. $K : 3x^2 + 3y^2 + 6xy - 6x + 12y = 0$



CENTRES OF PARTIALLY TRANSTABLE CONIC SECTIONS

Centre – Partially Transtable Conic Sections

Lemma

If there is a centre C of a conic section $K : ax^2 + by^2 + cxy + dx + ey + f = 0$, then it is in the form:

$$C = \left[\frac{ce - 2bd}{4ab - c^2}; \frac{cd - 2ae}{4ab - c^2}\right].$$

Theorem

The set of all centers of partially transstable conic sectios with respect to the given coefficients is ***a conic section*** *except in the cases of partial transtabilities with respect to x^2 and xy or y^2 and xy or xy and x or xy and y, i.e.* ***there are conic sectios of centers for the 11 partial transtabilities****.*

Proof:

The proof is divided into **15** parts, for each partial transtability. Consider a conic section in the form $ax^2 + by^2 + cxy + dx + ey + f = 0$. Then

Centre – Transtability with respect to x^2 and y^2

By complex calculations and verification using **GeoGebra** and **Wolfram Mathematica** software, we claim that the set of centres C_{x^2,y^2} is in the form:

$$\left(c^2 - 4ab\right) xy + cdx + cey + de = 0.$$

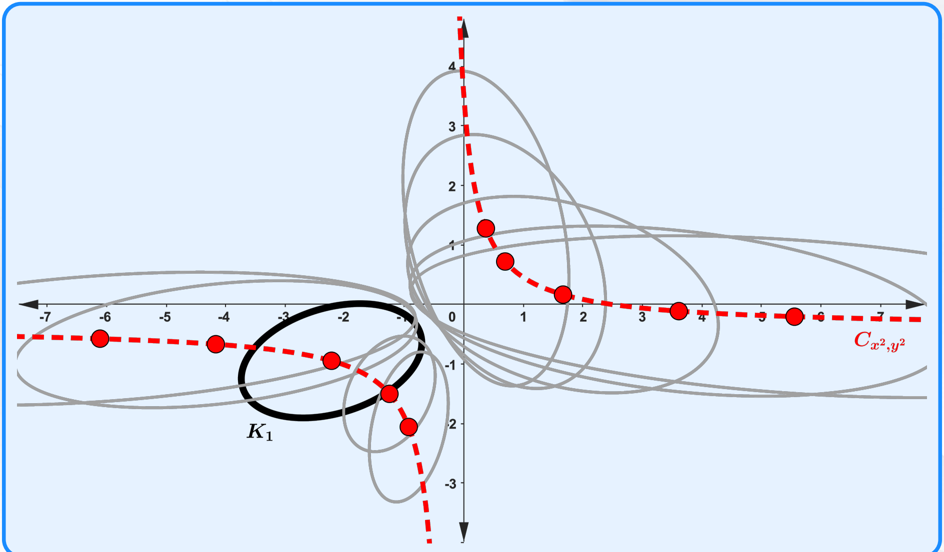
For Example

1. $K : \mathbf{2x^2 + 3y^2 - 3xy - 6x + 9y - 9 = 0}$

The Conic Section of Centres:

$$C_{x^2,y^2} : \mathbf{-36xy - 14x - 10y + 35 = 0}$$

Centre – Transtability with respect to x^2 and y^2

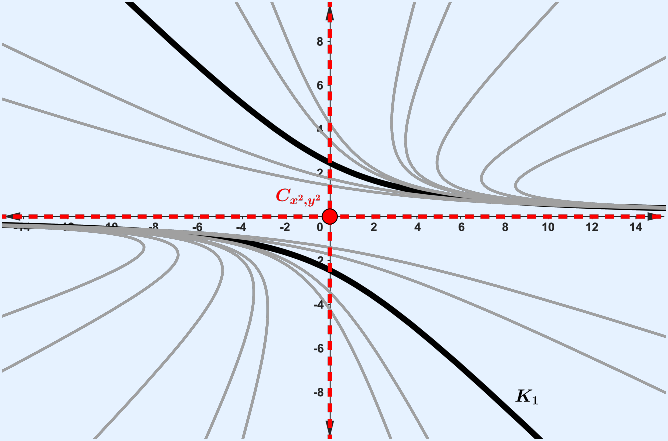


Centre – Transtability with respect to x^2 and y^2

For Example

2. $L : y^2 + xy - 6 = 0$

The Conic Section of Centres: $xy = 0$



Centre – Transtability with respect to x^2 and xy

By complex calculations and verification using **GeoGebra** and **Wolfram Mathematica** software, we claim that the set of centres $C_{x^2,xy}$

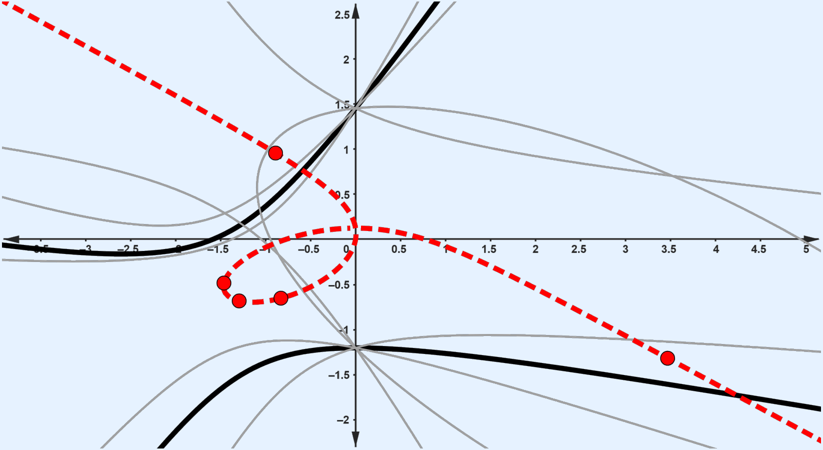
does not form any conic section.

This set of centres is in the form of a **„node“**.

Centre – Transtability with respect to x^2 and xy

For Example

3. $K : 4x^2 - 5y^2 + xy - 8x + 3y = 0$



CENTRE – TRANSTABILITY WITH RESPECT TO x^2 AND x

By complex calculations and verification using **GeoGebra** and **Wolfram Mathematica** software, we claim that the set of centres $C_{x^2,x}$ is in the form:

$$cx + 2by + e = 0.$$

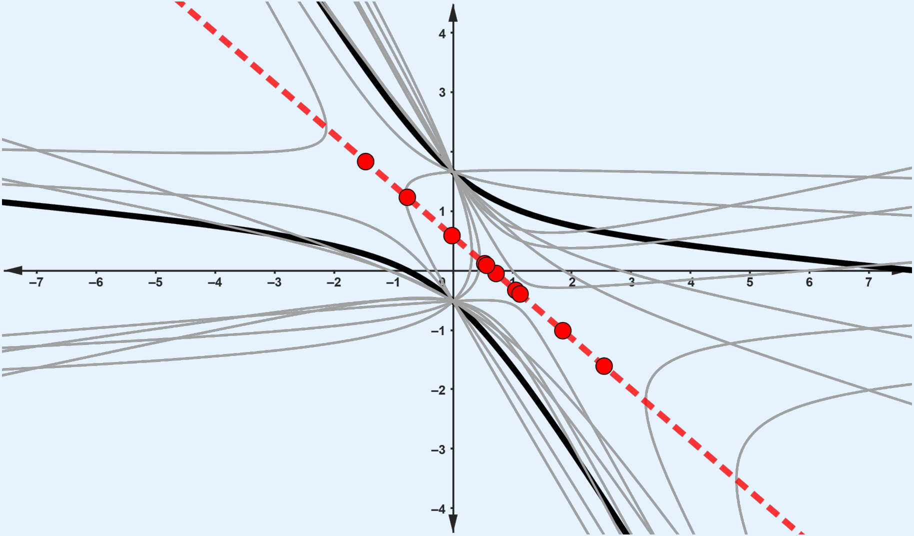
FOR EXAMPLE

4. $K : \boldsymbol{x^2 + 7y^2 + 12xy - 7x - 8y - 6 = 0}$

THE CONIC SECTION OF CENTRES:

$$C_{x^2,x} : \boldsymbol{12x + 14y - 8 = 0}$$

Centre – Transtability with respect to x^2 and x



Centre – Transtability with respect to x^2 and y

By complex calculations and verification using **GeoGebra** and **Wolfram Mathematica** software, we claim that the set of centres $C_{x^2,y}$ is in the form:

$$2bcy^2 + c^2xy + (cd - 2ae)\,x + 2bdy = 0.$$

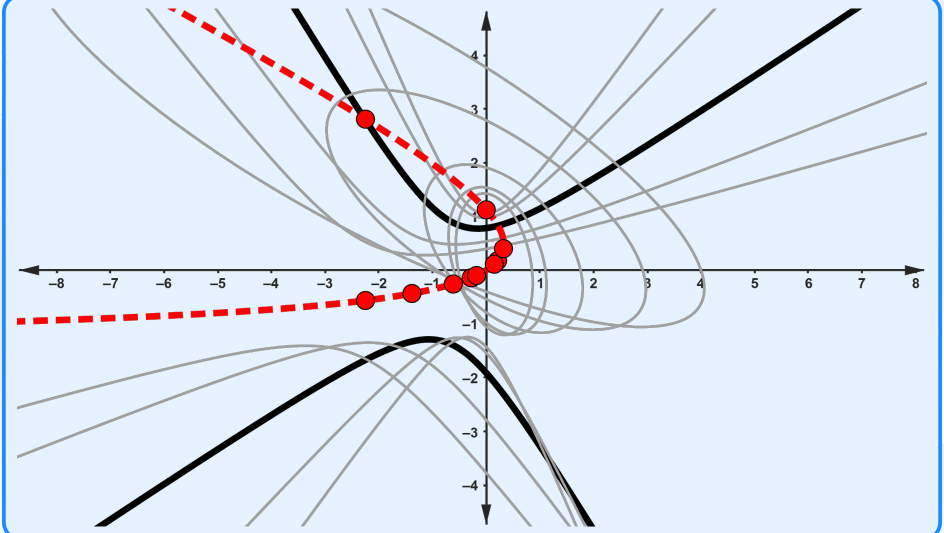
For Example

5. $K : \mathbf{9x^2 - 8y^2 - 8xy + 9x - 9y + 12 = 0}$

The Conic Section of Centres:

$$C_{x^2,y} : \mathbf{128y^2 + 64xy + 90x - 144y = 0}$$

CENTRE – TRANSTABILITY WITH RESPECT TO x^2 AND y



Centre – Transtability with respect to x^2 and $const$

By complex calculations and verification using **GeoGebra** and **Wolfram Mathematica** software, we claim that the set of centres $C_{x^2,const}$ is in the form:

$$cx + 2by + e = 0, \text{ i.e. } C_{x^2,const} = C_{x^2,x}.$$

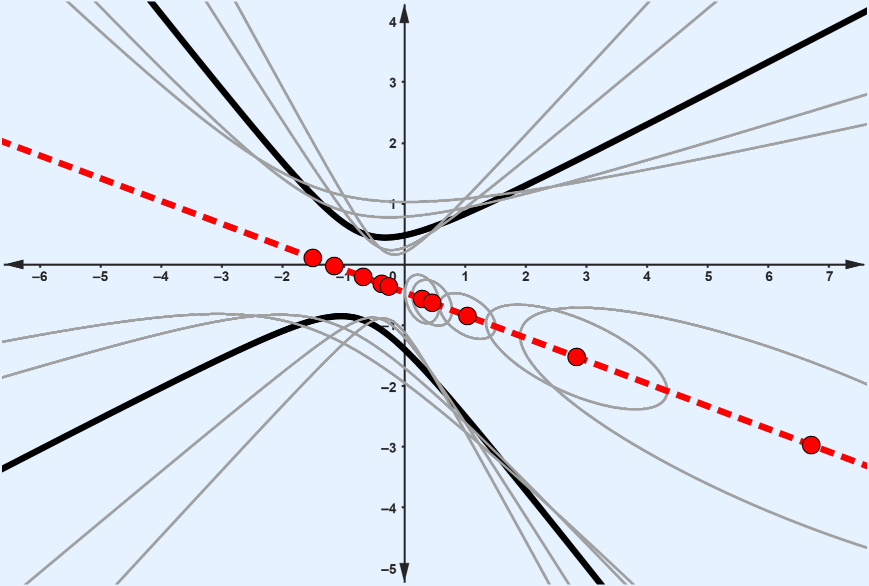
For Example

6. $K : \mathbf{8x^2 - 12y^2 - 9xy + 9x - 11y + 8 = 0}$

The Conic Section of Centres:

$$C_{x^2,const} : \mathbf{-9x - 24y - 11 = 0}$$

Centre – Transtability with respect to x^2 and $const$



CENTRE – TRANSTABILITY WITH RESPECT TO xy AND x

By complex calculations and verification using **GeoGebra** and **Wolfram Mathematica** software, we claim that the set of centres $C_{xy,x}$

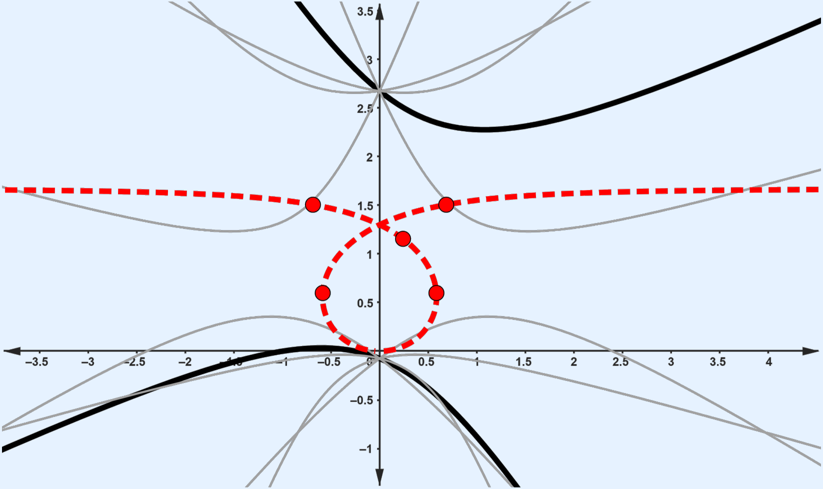
does not form any conic section.

This set of centres is in the form of a **„node“**.

CENTRE – TRANSTABILITY WITH RESPECT TO xy AND x

FOR EXAMPLE

7. $K : 10x^2 - 7y^2 - 3xy + 8x + 14y = 0$



CENTRE – TRANSTABILITY WITH RESPECT TO x AND y

By complex calculations and verification using **GeoGebra** and **Wolfram Mathematica** software, we claim that the set of centres $C_{x,y}$ is in the form:

$$2acx^2 + 2bcy^2 + \left(c^2 + 4ab\right)xy - de = 0.$$

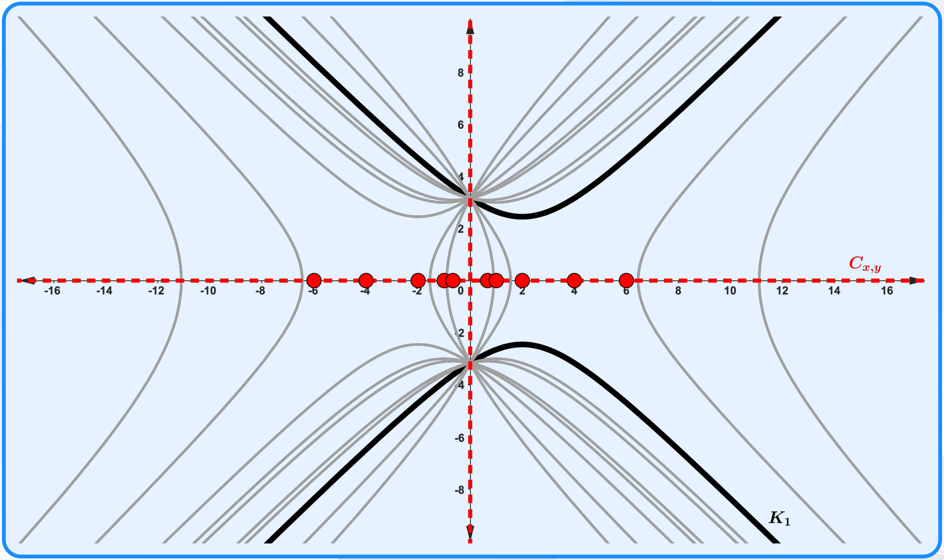
FOR EXAMPLE

8. $K : -\boldsymbol{x^2} + \boldsymbol{y^2} + \boldsymbol{4x} - \boldsymbol{10} = \boldsymbol{0}$

THE CONIC SECTION OF CENTRES:

$$C_{x,y} : \boldsymbol{-4xy = 0}$$

Centre – Transtability with respect to x and y



Centre – Transtability with respect to xy and $const$

By complex calculations and verification using **GeoGebra** and **Wolfram Mathematica** software, we claim that the set of centres $C_{xy,const}$ is in the form:

$$2ax^2 - 2by^2 + dx - ey = 0.$$

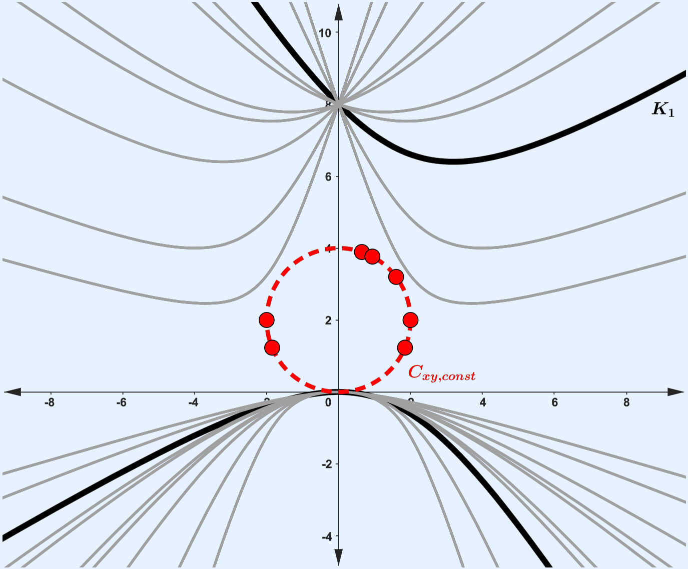
For Example

9. $K : \boldsymbol{x^2 - y^2 - xy + 8y = 0}$

The Conic Section of Centres:

$$C_{xy,const} : \boldsymbol{2x^2 + 2y^2 - 8y = 0.}$$

CENTRE – TRANSTABILITY WITH RESPECT TO xy AND $const$



Centre – Transtability with respect to x and $const$

By complex calculations and verification using **GeoGebra** and **Wolfram Mathematica** software, we claim that the set of centres $C_{x,const}$ is in the form:

$$cx + 2by + e = 0, \text{ i.e. } C_{x,const} = C_{x^2,x}.$$

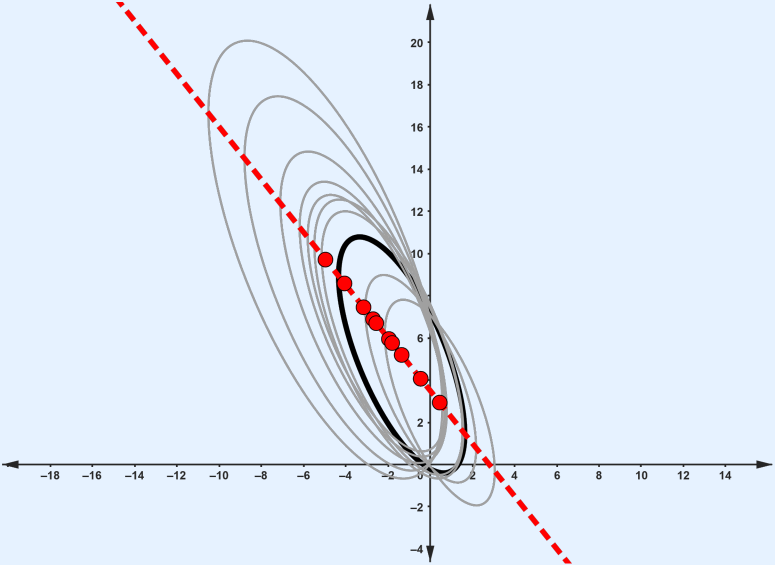
For Example

10. $K : \mathbf{7x^2 + 2y^2 + 5xy - 7x - 14y - 3 = 0}$

The Conic Section of Centres:

$$C_{x,const} : \mathbf{5x + 4y - 14 = 0}$$

Centre – Transtability with respect to x and $const$



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thank you!