# A Construction of Magmas and Related Representation

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# A simple construction

Magma $\mathbf{G} = (G, \cdot)$ is an algebra of type $(2)$.

Biunary algebra $\mathbf{I} = (I, \lambda, \rho)$ is an algebra of type $(1, 1)$.

Having $\mathbf{G}$ and $\mathbf{I}$, we define a magma $\mathbf{G}^{\mathbf{I}}$ via:

$$(x \cdot y)(i) := x(\lambda i) \cdot y(\rho i).$$

for any $x$, $y \in G$ and $i \in I$.

# Earlier results

If $\mathbf{G}$ is a semigroup and $\mathbf{I}$ is defined by the following table

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\lambda$ | 2 | 2 | 4 | 4 |
| $\rho$ | 3 | 4 | 3 | 4 |

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However, identities are not completely lost!
For instance, identity $axyb = ayxb$ holds. So $\mathbf{G}^{\mathbf{I}}$ is medial semigroup.

Moreover, the variety of all medial semigroups is generated by semigroups $\mathbf{G}^{\mathbf{I}}$, where $\mathbf{G}$ is a commutative semigroup.

# More general questions

Given an arbitrary but fixed magma $\mathbf{G}$ and biunary algebra $\mathbf{I}$, which identities are satisfied in $\mathbf{G}^{\mathbf{I}}$?

Given an arbitrary but fixed variety of magmas $\mathsf{M}$ and some ($\iota$-definable) variety $\mathsf{I}$ of biunary algebras, what identities axiomatize $V(\mathbf{G}^{\mathbf{I}} \mid \mathbf{G} \in \mathsf{M}, \mathbf{I} \in \mathsf{I})$?

# Categorical background – $\mathbf{G}^{\mathbf{I}}$

From now on, let Mag and BiUn denote the categories of magmas and of biunary algebras, respectively.

The construction of $\mathbf{G}^{\mathbf{I}}$ behaves functorially with respect to morphisms, leading to the functors:

$-^{\mathbf{I}}$ is an endofunctor on Mag
$\mathbf{G}^{-}$ is a contravariant functor from BiUn to Mag

# Categorical background – another construction

Let $\mathsf{I}$ be a $\iota$-definable variety of biunary algebras and let $\mathbf{G}$ and $\mathbf{H}$ be magmas.

We define the biunary algebra $[\mathbf{G}, \mathbf{H}] \in \mathsf{I}$ as follows.
The elements $\Omega \in [\mathbf{G}, \mathbf{H}]$ are mappings $\Omega \colon \mathcal{F}_{\mathsf{I}}(\iota) \times \mathbf{G} \longrightarrow \mathbf{H}$ satisfying

$$\Omega(t(\iota), x \cdot y) = \Omega(\lambda t(\iota), x) \cdot \Omega(\rho t(\iota), y)$$

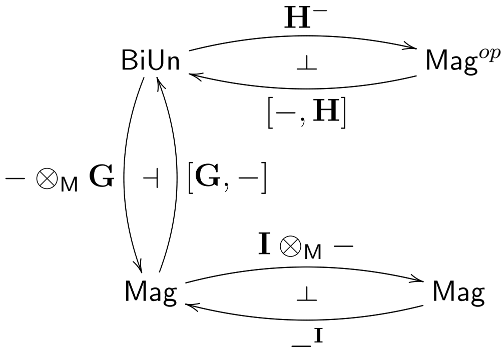
for any $x, y \in G$ and $t(\iota) \in \mathcal{F}_{\mathsf{I}}(\iota)$.

Further, for $\alpha \in \{\lambda, \rho\}$, we define

$$\alpha\Omega(t(\iota), x) = \Omega(t(\alpha\iota), x).$$

$[\mathbf{G}, -]$ is a functor from Mag to the category $\mathsf{I}$.
$[-, \mathbf{H}]$ is a contravariant functor from Mag to the category $\mathsf{I}$.

# Categorical background – adjoint pairs




$(\varepsilon,\eta)\colon \mathbf{H}^- \dashv [-,\mathbf{H}]$, where

$$\eta = (\eta_{\mathbf{G}}\colon \mathbf{G} \longrightarrow \mathbf{H}^{[\mathbf{G},\mathbf{H}]})_{\mathbf{G}\in\mathbf{Mag}},$$

$$(\eta_{\mathbf{G}}(x))(\Omega) = \Omega(\iota, x)$$

# Varieties and equalities

Let $X^\bullet$ denote the set of all magma terms over the set $X$.

**Theorem**

Let $\mathbf{H}$ be a magma and $X$ be a set. Then the kernel of the homomorphism:

$$\eta_{X^\bullet} : X^\bullet \longrightarrow \mathbf{H}^{[X^\bullet, \mathbf{H}]}$$

is a fully invariant congruence.

# Linearisation of terms

Let $X^\ell \subseteq X^\bullet$ denote the set of all linear terms (i.e., no variable repeats).

We denote $\mathrm{LIN} := \mathbb{N}^\ell$. Let $\ell\colon X^\bullet \longrightarrow \mathrm{LIN}$ denote a linearisation.

Linearisation numbers the positions of a term left-to-right.
For example,

$$\ell[x \cdot (y \cdot y)] = 1 \cdot (2 \cdot 3),$$

$$\ell[(x \cdot x) \cdot (y \cdot x)] = (1 \cdot 2) \cdot (3 \cdot 4).$$

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Further, for $\overline{x} \in X^\bullet$, the symbol $s[\overline{x}]$ denotes a mapping $\mathbb{N} \longrightarrow X$ telling which $x \in X$ sits at position $i$.

For any term $\overline{x} \in X^\bullet$, it holds that $\overline{x} = \ell[\overline{x}](s[\overline{x}](1), \ldots, s[\overline{x}](m))$.

For any term $\overline{x}$, each position can be represented by an $\iota$-biunary term.
For instance, if $\overline{x} = (x_1 \cdot x_1) \cdot x_2 \in G^{\mathcal{F}_{\mathsf{BiUn}}(\iota)}$, then

$$((x_1 \cdot x_1) \cdot x_2)(\iota) = (x_1 \cdot x_1)(\lambda\iota) \cdot x_2(\rho\iota) = (x_1(\lambda\lambda\iota) \cdot x_1(\rho\lambda\iota)) \cdot x_2(\rho\iota).$$

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For any $\overline{y} \in LIN_n$ and $i \in \{1, \ldots, n\}$, we denote $\iota$-term corresponding to position $i$ by $[\overline{y}, i](\iota)$.

$$\ell(\overline{x}) = (1 \cdot 2) \cdot 3,$$
$$[\ell(\overline{x}), 1](\iota) = \lambda\lambda\iota, \quad [\ell(\overline{x}), 2](\iota) = \rho\lambda\iota, \quad [\ell(\overline{x}), 3](\iota) = \rho\iota.$$

We use biunary $\iota$-terms to describe each individual position in the arbitrary magma term. (The linearization is just an auxiliary construction to make the formal description possible.)

Let $\overline{x}$ be a term in $n$ variables and let $h_1, \ldots, h_n$ be elements of $\mathbf{H}$. Then we use the following labeling:

$$\overline{x}_{i=1}^{n}(h_i) := \overline{x}^{\mathbf{H}}(h_1, \ldots, h_n).$$

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## Theorem

An arbitrary mapping $\Omega\colon \mathcal{F}_{\mathsf{l}}(\iota) \times X \longrightarrow \mathbf{H}$ induces $\overline{\Omega} \in [X^{\bullet}, \mathbf{H}]$ by the stipulation

$$\overline{\Omega}(t(\iota), \overline{x}) = \ell[\overline{x}]_{i=1}^{n}(\Omega([\ell[\overline{x}], i](t(\iota)), s[\overline{x}](i))).$$

Moreover, any element $\overline{\Omega} \in [X^{\bullet}, \mathbf{H}]$ is uniquely determined by its restriction $\Omega = \overline{\Omega} \restriction (\mathcal{F}_{\mathsf{l}}(\iota) \times X)$.

$$\overline{\Omega}(\iota, \overline{x}) = \ell[\overline{x}]_{i=1}^{n}(\Omega(\iota\text{-term for position } i,\ x \in X \text{ sitting at } i)).$$

# Towards the main theorem

Ingredients: $\iota$-definable variety of biunary algebras $\mathsf{I}$, some variety of magmas $\mathsf{Mag}$.
Goal: Axiomatization of $V(\mathbf{H}^{\mathbf{I}} \mid \mathbf{H} \in \mathsf{V}, \mathbf{I} \in \mathsf{I})$.

Ingredients: $\iota$-definable variety of biunary algebras I, some variety of magmas Mag.
Goal: Axiomatization of $V(\mathbf{H}^{\mathbf{I}} \mid \mathbf{H} \in \mathsf{V}, \mathbf{I} \in \mathsf{I})$.

(1) Let $\mathbf{H} \in \mathsf{M}$. We know that for $\eta_{X^\bullet} \colon X^\bullet \longrightarrow \mathbf{H}^{[X^\bullet, \mathbf{H}]}$, the kernel $\mathrm{Ker}(\eta_{X^\bullet})$ is fully invariant, meaning that the algebra $X^\bullet/(\mathrm{Ker}(\eta_{X^\bullet}))$ is a free algebra of some variety.
At this point, we only conjecture that this variety is $V(\mathbf{H}^{\mathbf{I}} \mid \mathbf{H} \in \mathsf{V}, \mathbf{I} \in \mathsf{I})$.

(2) Let $(\overline{x}_{i=1}^{m}(x_i), \overline{y}_{i=1}^{n}(y_i)) \in \mathrm{Ker}(\eta_{X^\bullet})$ and let $\Omega \colon \mathcal{F}_{\mathsf{I}}(\iota) \times X \longrightarrow \mathbf{H}$ be an arbitrary mapping.
There is a unique $\overline{\Omega} \in [X^\bullet, \mathbf{H}]$ such that:

$$(\overline{x}^{\mathbf{H}})_{i=1}^{m}(\Omega([\overline{x}, i]\iota, x_i)) = \overline{\Omega}(\iota, \overline{x}_{i=1}^{m}(x_i)) = (\eta_{X^\bullet}(\overline{x}_{i=1}^{m}(x_i)))(\overline{\Omega})$$
$$= (\eta_{X^\bullet}(\overline{y}_{i=1}^{m}(y_i)))(\overline{\Omega}) = (\overline{y}^{\mathbf{H}})_{i=1}^{m}(\Omega([\overline{y}, i]\iota, y_i)).$$

# Towards the main theorem

(3)

$$(\overline{x}^{\mathbf{H}})_{i=1}^{m}(\Omega(\underbrace{[\overline{x}, i]\iota}_{\iota\text{-term for position of } x_i}, x_i)) = (\overline{y}^{\mathbf{H}})_{i=1}^{m}(\Omega(\underbrace{[\overline{y}, i]\iota}_{\iota\text{-term for position of } y_i}, y_i))$$

holding for any $\mathbf{H} \in \mathsf{M}$ yields existence of an identity $\overline{x}_{i=1}^{m}(x_i') \approx \overline{y}_{i=1}^{n}(y_i')$ satisfying:

(i) $\mathsf{M} \vDash \overline{x}_{i=1}^{m}(x_i') \approx \overline{y}_{i=1}^{n}(y_i')$,

(ii) $x_i' = x_j'$ if and only if $(x_i = x_j$ and $[\overline{x}, i](\iota) \approx [\overline{x}, j](\iota))$,

(iii) $x_i' = y_j'$ if and only if $(x_i = y_j$ and $[\overline{x}, i](\iota) \approx [\overline{y}, j](\iota))$,

(iv) $y_i' = y_j'$ if and only if $(y_i = y_j$ and $[\overline{y}, i](\iota) \approx [\overline{y}, j](\iota))$.

Identities holding in M are "restricted" via identities holding in I.

## Main theorem

For every variety M of magmas and every $\iota$-definable variety I of biunary algebras, the variety $\mathcal{V}(\mathbf{H}^{\mathbf{I}} \mid \mathbf{H} \in \mathsf{M}, \mathbf{I} \in \mathsf{I})$ is axiomatized by the identities

$$\overline{x}^m_{i=1}(x_i) \approx \overline{y}^n_{i=1}(y_i),$$

where $\overline{x} \in LIN_m$ and $\overline{y} \in LIN_n$ satisfy the following conditions:

(i) $\mathsf{M} \models \overline{x}^m_{i=1}(x_i) \approx \overline{y}^n_{i=1}(y_i)$,

(ii) If $x_i = x_j$, then $\mathsf{I} \models [\overline{x}, i](\iota) \approx [\overline{x}, j](\iota)$,

(iii) If $x_i = y_i$, then $\mathsf{I} \models [\overline{x}, i](\iota) \approx [\overline{y}, i](\iota)$,

(iv) If $y_i = y_j$, then $\mathsf{I} \models [\overline{y}, i](\iota) \approx [\overline{y}, j](\iota)$.

## Theorem

$$V(\mathbf{H}^{\mathbf{I}} \mid \mathbf{H} \in \mathsf{M}, \mathbf{I} \in \mathsf{I}) = V(\mathbf{H}^{\mathcal{F}_{\mathsf{I}}(\iota)} \mid \mathbf{H} \in \mathsf{M}).$$

# Examples

Let V $= V(\mathbf{H}^{\mathbf{I}} \mid \mathbf{H} \in$ M, $\mathbf{I} \in$ I).

| M | I | V satisfies | V belongs to |
|---|---|---|---|
| commutative semigroups | $\lambda \approx \lambda\rho$ | $x(yz) \approx y(xz)$ | left permutable grupoids |
| commutative semigroups | $\rho \approx \lambda\lambda$, $\lambda \approx \lambda\rho$ | $(xy)z \approx (zy)x$, $x(yz) \approx y(xz)$ | AG**-grupoids |
| idempotent magmas | $\lambda \approx \lambda\lambda$ $\lambda \approx \rho\lambda$ | $(xx)y \approx xy$ | left idempotent grupoids |

# References


- Botur, M.: *On semigroup constructions induced by commuting retractions on a set*, Algebra Univer. **82**, 62 (2021)
- Botur, M., Kowalski, T. : *Beyond wreath and block*, Semigroup Forum, **105** (1), 96–116 (2022)
- Nagy, A.: *Special Classes of Semigroups*, Springer (2001)
- Strecker, R.: *Construction of medial semigroups*. Commentat. Math. Univ. Carol. **25**, 689–697 (1984)