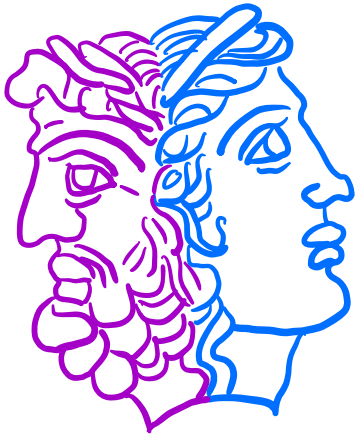
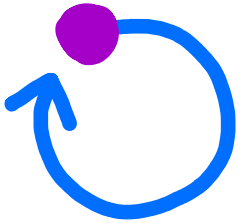
JANUS-FACES OF TEMPORAL CSPs
A DICHOTOMY OF EXPRESSIVITY



Johanna Brunar
TU Wien

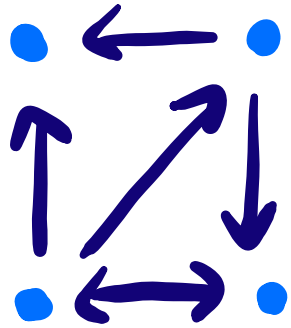


joint work with M. Pinsker, M. Schöbi

SSAOS'25, Sep. 2025

CONSTRAINT SATISFACTION PROBLEMS

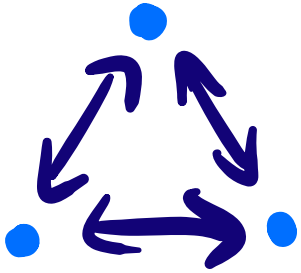
$\mathbb{A} = (A; (R_i)_{i \in I})$ RELATIONAL STRUCTURE WITH SIGNATURE τ



CSP($\mathbb{A}$):

- INPUT: $\mathbb{X}$ FINITE τ-STRUCTURE
- QUESTION: DOES THERE EXIST A HOMOMORPHISM $\mathbb{X} \to \mathbb{A}$?

CONSTRAINT SATISFACTION PROBLEMS

$\mathbb{A}$	$\mathbb{X} \to \mathbb{A}$?
	3-COLOURABLE ?
$(\mathbb{Q}; <)$	ACYCLIC ?
$(\mathbb{Z}; \{0\}, \{1\}, +, -)$	EQUATION SOLVABLE?

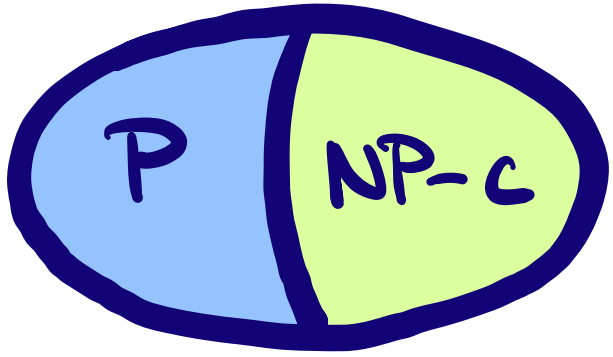
CONSTRAINT SATISFACTION PROBLEMS

FINITE-DOMAIN CSP-DICHOTOMY:

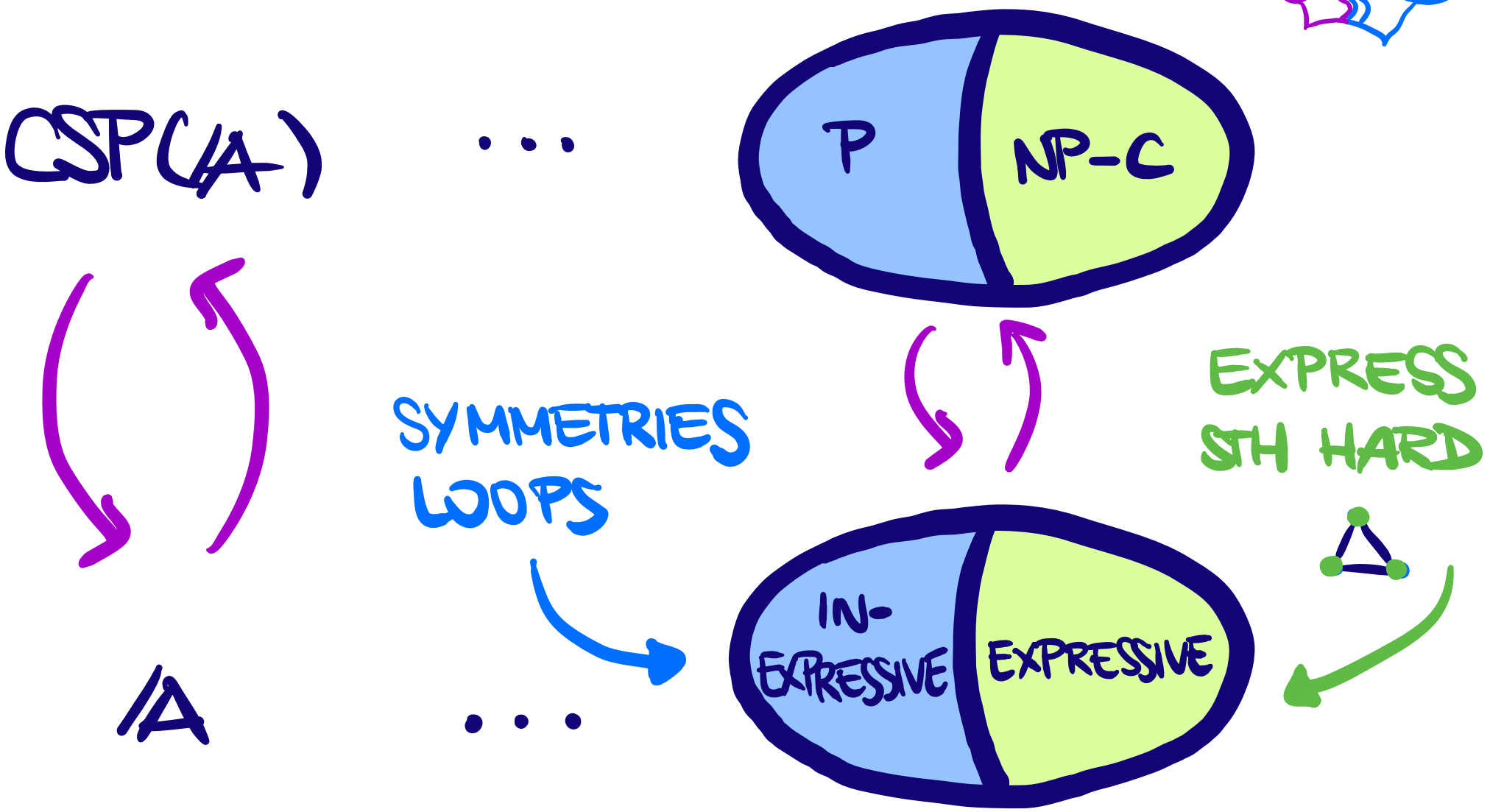
THM (BULATOV, ZHUK '17)

$\mathbb{A}$ FINITE.

CSP($\mathbb{A}$) IS SOLVABLE IN P-TIME OR NP-COMPLETE.



ALGEBRAIC APPROACH

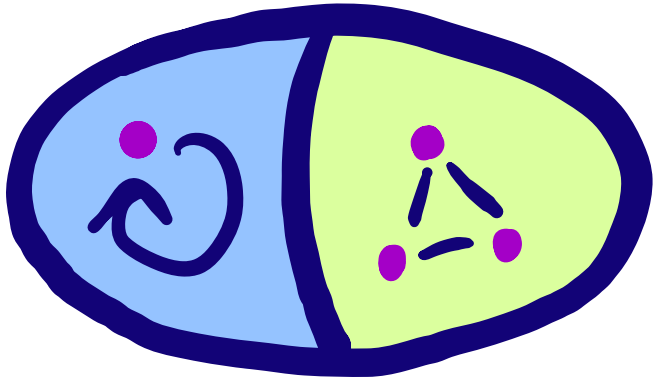


ALGEBRAIC APPROACH

THM (HELL+NEŠETŘIL '90)

$\mathbb{G}$ NON-BIPARTITE FINITE GRAPH.
EITHER $\mathbb{G}$ IS OMNI-EXPRESSIVE
OR $\mathbb{G}$ HAS A LOOP.

CSP($\mathbb{G}$)
TRIVIAL



CSP($\mathbb{G}$)
NP-C

ALGEBRAIC APPROACH

THM (BULATOV, ZHUK '17, KEARNES + MARKOVIC + MCKENZIE '15)

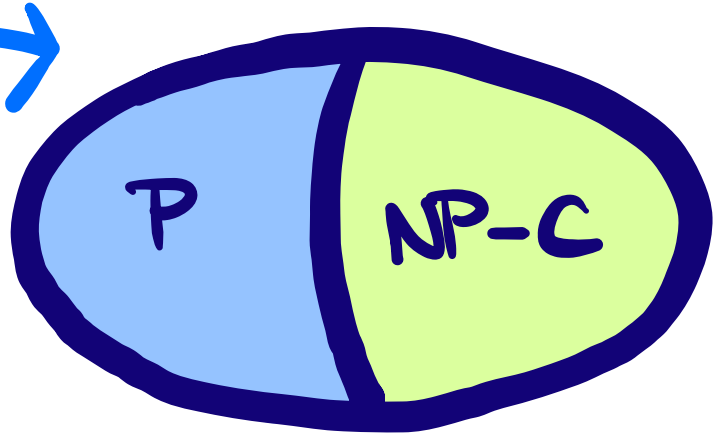
$\mathbb{A}$ FINITE.

EITHER $\mathbb{A}$ IS OMNI-EXPRESSIVE & CSP($\mathbb{A}$) NP-C

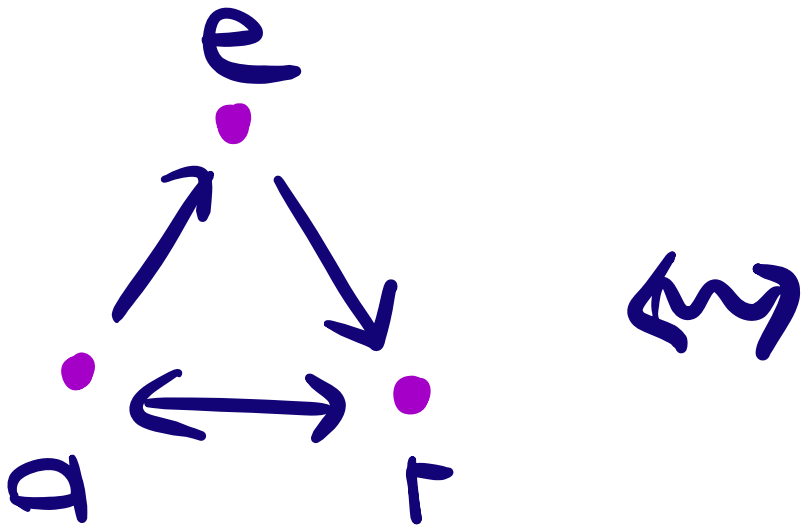
OR $Pol(\mathbb{A}) \models$ SIGGERS & CSP($\mathbb{A}$) IN P.

$\forall a, r, e:$

$$s(a, r, e, a) = s(r, a, r, e)$$



LOOPS ⇝ IDENTITIES



$\rightsquigarrow$

$$s(a,r,e,a)$$
$$\downarrow\ \downarrow\ \downarrow\ \downarrow$$
$$s(r,a,r,e)$$

$$F = \left\langle \binom{a}{r}\binom{r}{a}\binom{e}{r}\binom{a}{e} \right\rangle_{Pol(A)}$$

IS OMNI-EXPRESSIVE OR HAS LOOP.

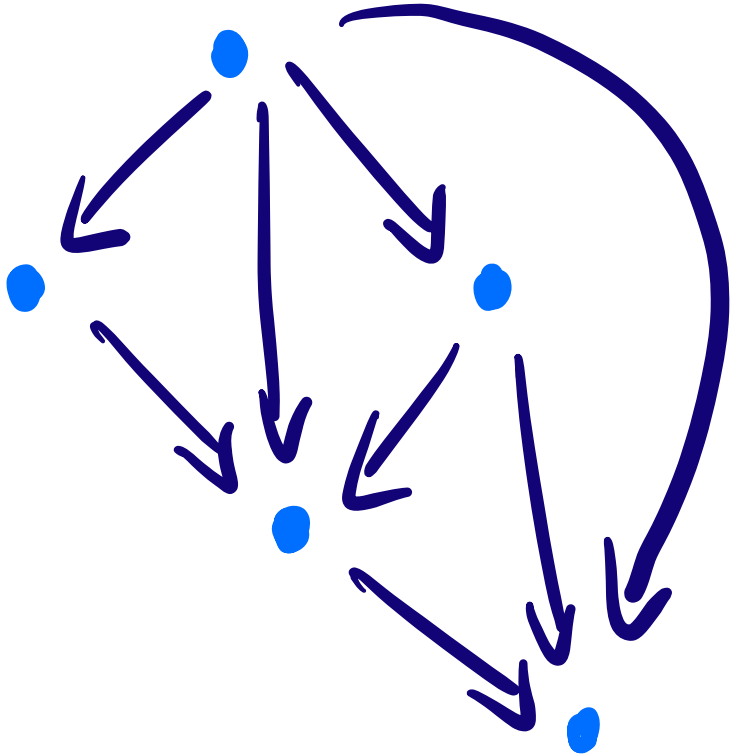

FROM FINITE
TO INFINITE

FROM FINITE TO INFINITE

CSP($\mathbb{Q}$;<)

INPUT: $\mathbb{X}$ FINITE DIGRAPH

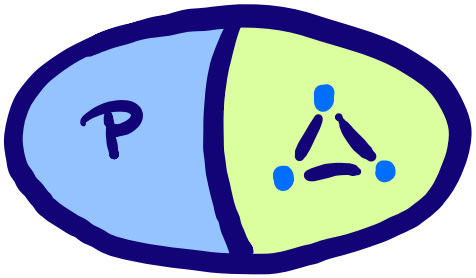
QUESTION: IS $\mathbb{X}$ ACYCLIC?



FROM FINITE TO INFINITE

TEMPORAL CONSTRAINT LANGUAGE (TCL)
= F.O. REDUCT OF $(\mathbb{Q}; <)$

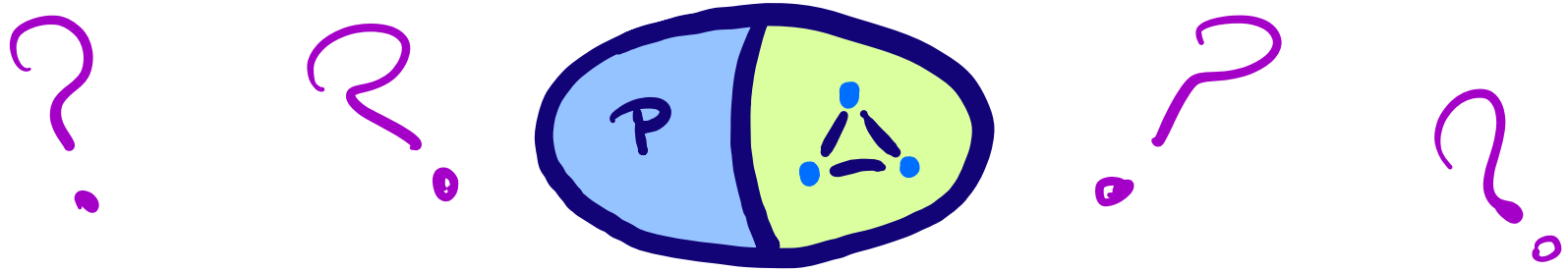
THM (BODIRSKY + KÁRA '08)
$\mathbb{A}$ TCL.
EITHER $\mathbb{A}$ IS OMNI-EXPRESSIVE OR
CSP($\mathbb{A}$) IS SOLVABLE IN P-TIME.



FROM FINITE TO INFINITE

CONJ: (BODIRSKY + PINSKER '12)

$\mathbb{A}$ F.O. REDUCT OF A FINETELY BOUNDED HOMOGENEOUS STRUCTURE. THEN $CSP(\mathbb{A})$ IS SOLVABLE IN P-TIME OR $\mathbb{A}$ IS OMNI-EXPRESSIVE.



CLOSE TO FINITE: ω-CATEGORICITY

$\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$

$\bar{a} \sim_n \bar{b} :\Leftrightarrow \exists \alpha \in \mathrm{Aut}(\mathbb{A}) : \alpha(\bar{a}) = \bar{b}$

$\mathbb{A}$ IS ω-CATEGORICAL IFF $\sim_n$ HAS FINITELY MANY CLASSES FOR ALL n.

EX $\mathbb{A} = (\mathbb{Q}; <)$

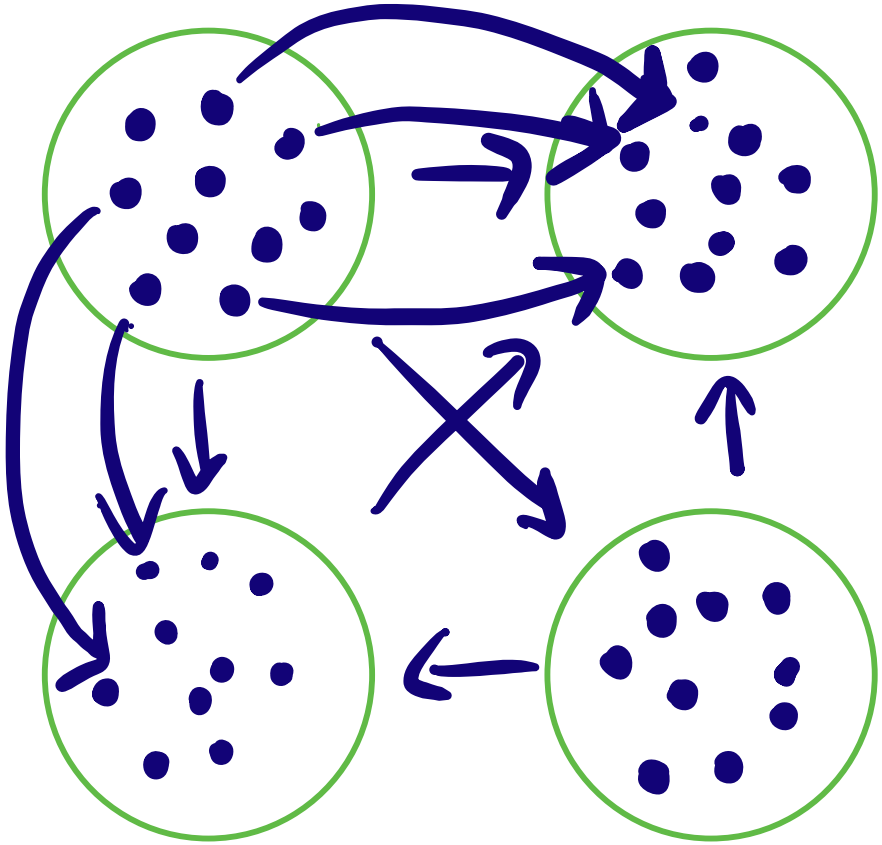
FINITELY MANY ORDERS ON n-TUPLES!

CLOSE TO FINITE: ω-CATEGORICITY

$\mathbb{E}$ ω-CATEGORICAL DIGRAPH

FINITE FACTOR:

$\mathbb{E}/\mathrm{Aut}(\mathbb{E})$



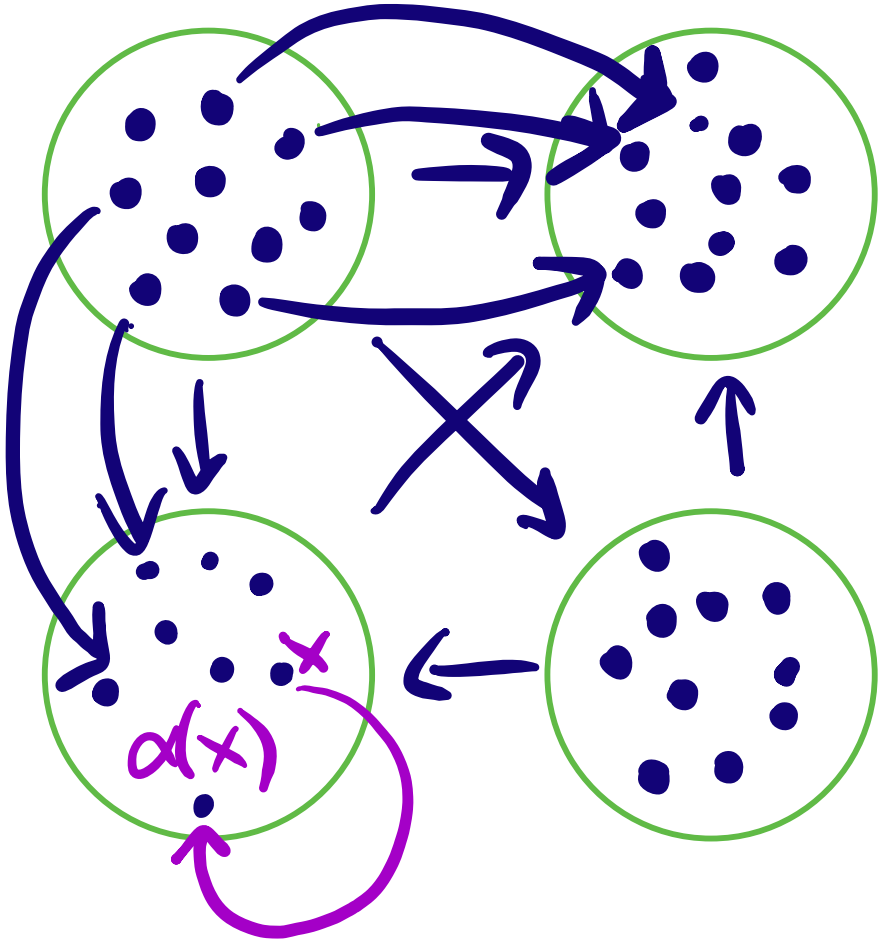
CLOSE TO FINITE: ω-CATEGORICITY

$\mathbb{E}$ ω-CATEGORICAL DIGRAPH

FINITE FACTOR:

$\mathbb{E}/\mathrm{Aut}(\mathbb{E})$

PSEUDO-LOOP



INFINITE HELL-NESS

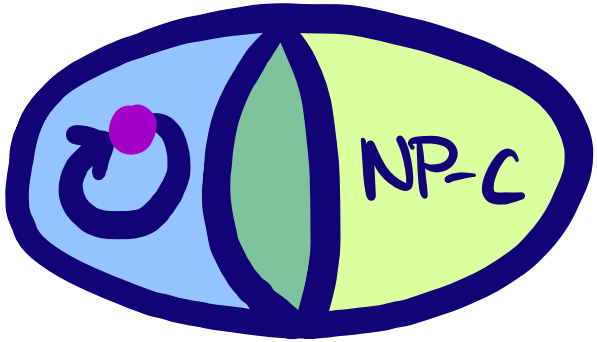
Thm (Barto + Bodor + Kozik + Mottet + Pinsker '23)

$\mathbb{G}$ smooth, ω-categorical

$\mathbb{G}/\mathrm{Aut}(\mathbb{G})$ SYMMETRIC, NON-BIPARTITE.

EITHER $\mathbb{G}$ IS OMNI-EXPRESSIVE

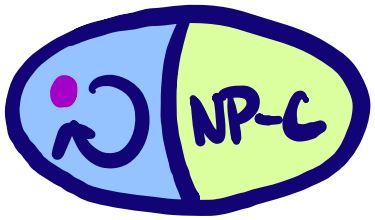
OR $\mathbb{G}/\mathrm{Aut}(\mathbb{G})$ HAS PSEUDO-LOOP.



A DICHOTOMY OF EXPRESSIVITY

Thm (B.+PINSKER+SCHÖBI '25)

$\mathbb{A}$ TCL.



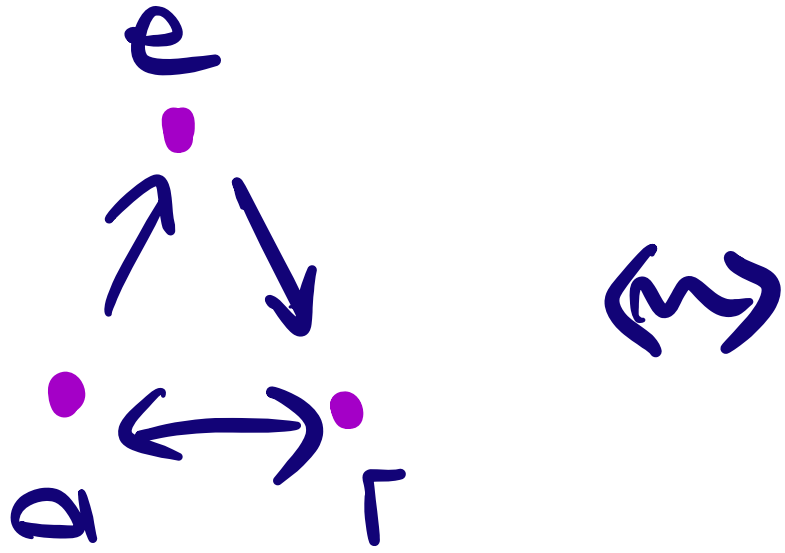
EITHER $\mathbb{A}$ IS OMNI-EXPRESSIVE

OR IT CAN ONLY EXPRESS

DIGRAPHS* WITH PSEUDO-LOOPS.

* SMOOTH, ALGEBRAIC LENGTH 1

PSEUDO-LOOPS ⇝ PSEUDO-IDENTITIES



$\rightsquigarrow$

$$u \circ s(a, r, e, a)$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$v \circ s(r, a, r, e)$$

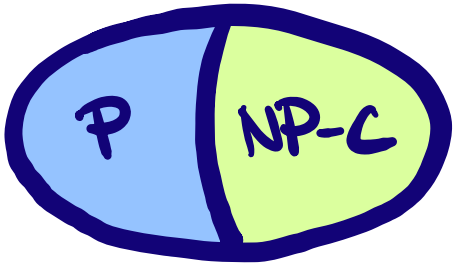
$$\mathcal{F} = \left\langle \begin{pmatrix} a & r & e & a \\ r & a & r & e \end{pmatrix} \right\rangle_{PSI(A)}$$

OMNI-EXPRESSIVE OR HAS PSEUDO-LOOP.

A DICHOTOMY OF EXPRESSIVITY

Thm (B.+PINSKER+SCHÖBI '25)

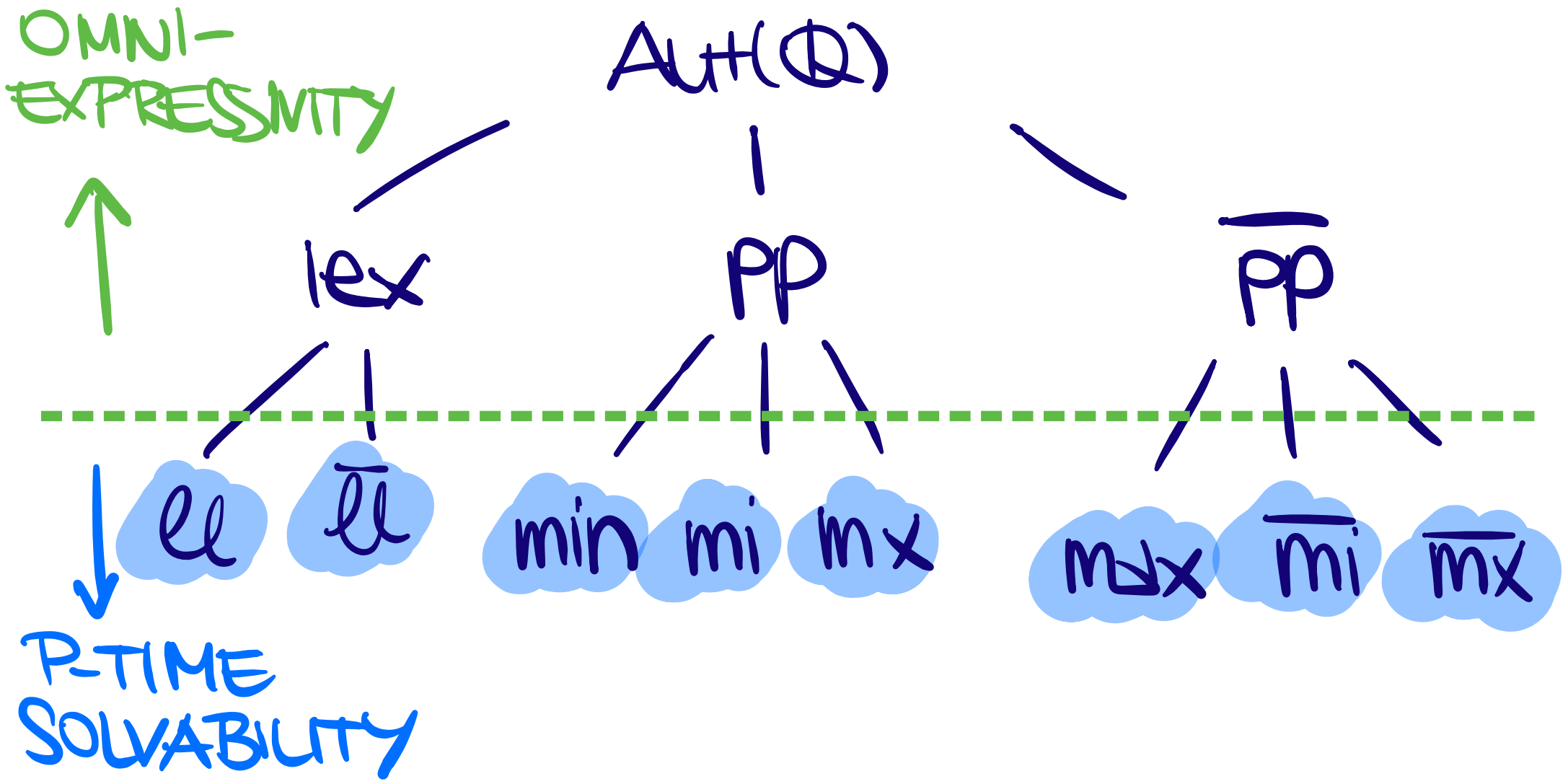
$\mathbb{A}$ TCL.



EITHER $\mathbb{A}$ IS OMNI-EXPRESSIVE

OR $\mathrm{Pol}(\mathbb{A}) =$ PSEUDO-SIGGERS.

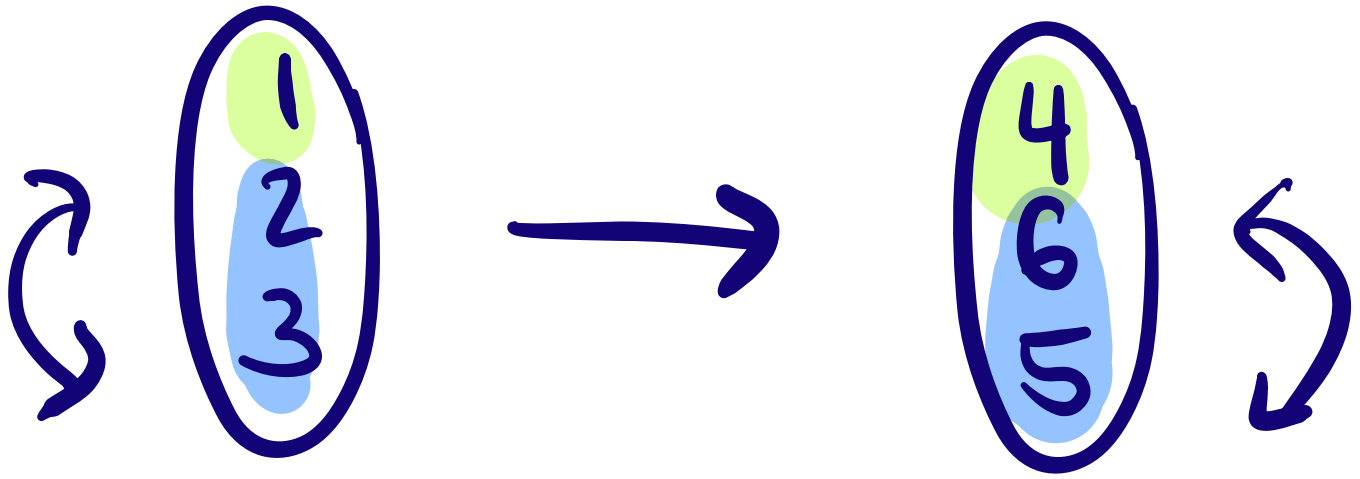
SNEAK PEEK: PROOF



SNEAK PEEK: PROOF

G DIGRAPH ON $\mathbb{Q}^k$

INDUCTION ON k



APPLY POLYMORPHISMS

WHY DO WE CARE?

- PRACTICAL IMPORTANCE: AI
- POORLY UNDERSTOOD

ALGORITHMS

DESCRIPTIVE COMPLEXITY

ALGEBRA

WHY DO WE CARE?

QUESTION $\mathbb{A}$ ω-CAT.

IS IT TRUE THAT EITHER
$\mathbb{A}$ IS OMNI-EXPRESSIVE
OR $\mathrm{Pol}(\mathbb{A})$ = PSEUDO-SIGGERS ?

WHY DO WE CARE?

- PRACTICAL IMPORTANCE: AI
- POORLY UNDERSTOOD

ALGORITHMS

DESCRIPTIVE COMPLEXITY

ALGEBRA

- BEAUTIFUL!

When Janus can't express it all
He tries in vain, he hits a wall
When for IK3 no way he knows
His face of pseudo-boys he shows