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# Axiomatising Ex-lattices

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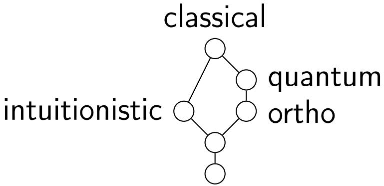
Propositional logics
in $\wedge, \vee, 0, 1, \neg$




W. Holliday (2023):

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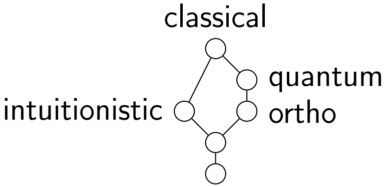


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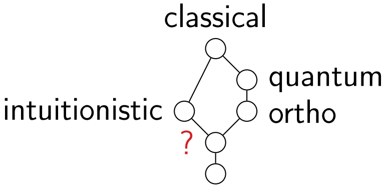


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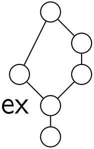

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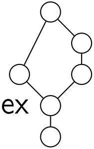


$$\mathcal{L} \models \varphi \vdash_{\mathsf{PC}} \psi$$

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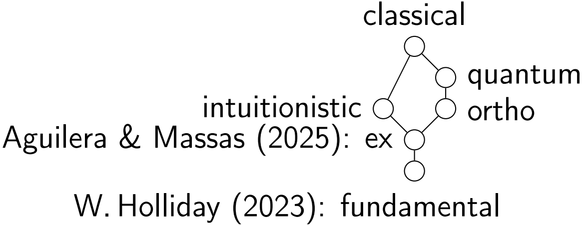
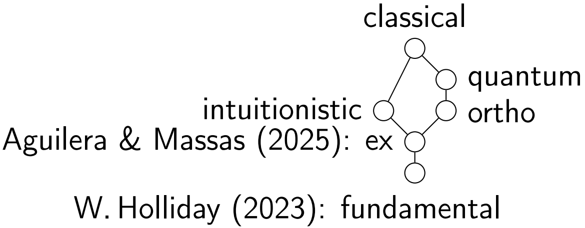


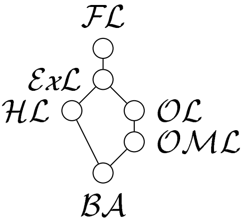

$\mathcal{L} \models \varphi \vdash_{\mathsf{PC}} \psi$ $\qquad\qquad$ $\mathcal{V} \models \varphi \lessapprox \psi$

Propositional logics in $\wedge, \vee, 0, 1, \neg$




$$\mathcal{L} \models \varphi \vdash_{\mathsf{PC}} \psi$$

Supervarieties of Boolean algebras




$$\mathcal{V} \models \varphi \lessapprox \psi$$

# Notational conventions

## Change of symbols

| | | |
|---|---|---|
| $\wedge$ | $x \wedge y \rightsquigarrow x \cdot y \rightsquigarrow xy$ | multiplication $\equiv$ conjunction |
| $\vee$ | $x \vee y \rightsquigarrow x + y$ | addition $\equiv$ disjunction |
| $\neg$ | $\neg(\ldots) \rightsquigarrow \overline{\quad \ldots \quad}$ | overline $\equiv$ negation |
| 0, 1 | no change | constants |
| precedence | $\overline{\phantom{x}} \prec \cdot \prec +$ | |

## Example

- $\neg(\neg x \wedge \neg y) \rightsquigarrow \overline{\overline{x} \cdot \overline{y}}$
- $\neg(x \wedge y) \vee (z \wedge y) \rightsquigarrow \overline{xy} + zy$

# Fundamental lattices: $\mathcal{FL}$

**Definition:** $\mathbf{L} = \langle L; \cdot, +, 0, 1, \bar{\ } \rangle$ is a fundamental lattice if

- $\langle L; \cdot, +, 0, 1 \rangle$ is a bounded lattice
- $\mathbf{L} \models \quad x\overline{x} \approx 0$ (semicomplementation)
- $\mathbf{L} \models \quad (\forall x\, y\colon (x \leq y \rightarrow \overline{y} \leq \overline{x}))$ (order reversal)
- $\mathbf{L} \models \quad x \lessapprox \overline{\overline{x}}$ (extensionality)

**Characterisation:** $\mathcal{FL}$ is a variety given by identities/inequalities

- bounded lattice equations
- $x\overline{x} \approx 0$
- $\overline{x} \lessapprox \overline{xy}$
- $x \lessapprox \overline{\overline{x}}$

# Quantum lattices: $\mathcal{OL}$ and $\mathcal{OML}$

**Definition:** $\mathcal{OL} \subseteq \mathcal{FL}$ is the subvariety given by adding

$\overline{\overline{x}} \approx x$ (involution)

**Definition:** $\mathcal{OML} \subseteq \mathcal{OL}$ is the subvariety given by adding

$x + y \lessapprox x + \overline{x}(x + y)$ (orthomodular inequality)

# Constructivist lattices: $\Psi\mathcal{FL}$ and $\mathcal{HL}$

**Definition:** $\Psi\mathcal{FL} \subseteq \mathcal{FL}$ is the subquasivariety given by adding

$\forall x, y\colon (xy = 0 \to y \leq \overline{x})$ (pseudocomplementation law)

**Characterisation:** $\Psi\mathcal{FL}$ is a variety given by

- bounded lattice equations
- $x\overline{x} \approx 0$
- $x\overline{y} \approx x\overline{x y}$
- $\overline{0} \approx 1$ (pseudocomplemented fundamental lattices)

**Characterisation:** $\Psi\mathcal{FL} \subseteq \mathcal{FL}$ is the subvariety given by adding

$x\overline{y} \approx x\overline{x y}$ (pseudocomplementation identity)

**Definition:** $\mathcal{HL} \subseteq \Psi\mathcal{FL}$ is the subvariety given by adding

$x(y+z) \approx xy + xz$ (distributivity)

# Ex-lattices: $\mathcal{ExL}$

**Def. (Aguilera & Massas):** $\mathcal{ExL} \subseteq \mathcal{FL}$ is the subvariety given by adding (ex):

$$\overline{x(yw+yu)} \cdot x(w+v) \cdot \overline{\overline{z}} \lessapprox \overline{\overline{xz}} \cdot (xw+xv+z) \cdot \left(y(w+u) + \overline{y(w+u)}\right)$$

# Ex-lattices: $\mathcal{E}x\mathcal{L}$

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Observation: $\mathcal{E}x\mathcal{L}$ is the variety given by

- fundamental lattice identities
- $\overline{x(yw+yu)} \cdot x(w+v) \cdot \overline{\overline{z}} \lessapprox \overline{\overline{xz}}$ (ex1)
- $\overline{x(yw+yu)} \cdot x(w+v) \cdot \overline{\overline{z}} \lessapprox xw+xv+z$ (ex2)
- $\overline{x(yw+yu)} \cdot x(w+v) \cdot \overline{\overline{z}} \lessapprox y(w+u) + \overline{y(w+u)}$ (ex3)

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**Lemma (Aguilera & Massas):** $\mathcal{E}x\mathcal{L}$ satisfy (4 variables!)

- $\overline{\overline{x}} \cdot \overline{\overline{y}} \lessapprox \overline{\overline{xy}}$ (nu)
- $x(y + z)\overline{\overline{w}} \lessapprox xy + xz + w$ (vi)
- $\overline{w(xy + xz)}w \lessapprox x(y + z) + \overline{x(y + z)}$ (cl)

# Aguilera & Massas prove:

**$\mathcal{OL} \vee \mathcal{HL} \subseteq \mathcal{ExL}$**

- $\mathcal{OL} \subseteq \mathcal{ExL}$
- $\mathcal{HL} \subseteq \mathcal{ExL}$
- hence $\mathcal{OL} \cup \mathcal{HL} \subseteq \mathcal{ExL}$

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**$\mathcal{E}x\mathcal{L} \subseteq \mathcal{OL} \vee \mathcal{HL}$**

- $\mathcal{E}x\mathcal{L} \subseteq \mathcal{N}u\mathcal{L} \cap \mathcal{V}i\mathcal{L} \cap \mathcal{C}l\mathcal{L} \subseteq \mathcal{FL}$

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- $\forall \mathbf{L} \in \mathcal{N}u\mathcal{L} \cap \mathcal{V}i\mathcal{L} \cap \mathcal{C}l\mathcal{L}$
  $\exists \mathbf{O} \in \mathcal{OL}\, \exists \mathbf{I} \in \mathcal{HL}\, \exists e\colon \quad e\colon \mathbf{L} \hookrightarrow \mathbf{O} \times \mathbf{I}$ subdirect embedding

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**$\mathcal{ExL} \subseteq \mathcal{OL} \vee \mathcal{HL}$**

- $\mathcal{ExL} \subseteq \mathcal{NuL} \cap \mathcal{ViL} \cap \mathcal{ClL} \subseteq \mathcal{FL}$
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- $\forall$ common identities $s \approx t$ of $\mathcal{OL}$ and $\mathcal{HL}$
  $\mathbf{O} \models s \approx t, \quad \mathbf{I} \models s \approx t, \quad \mathbf{O} \times \mathbf{I} \models s \approx t \qquad \rightsquigarrow \mathbf{L} \models s \approx t$

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**Theorem (Aguilera & Massas): $\mathcal{ExL} = \mathcal{OL} \vee \mathcal{HL} \subseteq \mathcal{FL}$**

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**Theorem (Aguilera & Massas): $\mathcal{ExL} = \mathcal{OL} \vee \mathcal{HL} \subseteq \mathcal{FL}$ is… (4 vars!)**

…the subvariety given by adding to $\mathcal{FL}$-axioms: (nu), (vi), (cl)

# Characterisation of (nu)

**Characterisation (MB): For $\mathbf{L} \in \mathcal{FL}$ TFAE**

- $\mathbf{L} \models$ (ex1) (ex1)
- $\mathbf{L} \models \overline{\overline{x}} \cdot \overline{\overline{y}} \approx \overline{\overline{xy}}$ (nu=)
- $\mathbf{L} \models \overline{\overline{x}} \cdot \overline{\overline{y}} \lessapprox \overline{\overline{xy}}$ (nu)
- $\mathbf{L} \models x \cdot \overline{\overline{y}} \lessapprox \overline{\overline{xy}}$ (nu≤)
- $\mathbf{L} \models x \cdot \overline{\overline{y}} \approx x\overline{\overline{xy}}$
- $\mathbf{L} \models \overline{x \cdot \overline{\overline{y}}} \approx \overline{xy}$ (*)
- $\mathbf{L} \models \overline{\overline{\overline{x}} \cdot \overline{\overline{y}}} \approx \overline{xy}$

# Characterisation of (vi)

**Characterisation (MB): For $\mathbf{L} \in \mathcal{FL}$ TFAE**

- $\mathbf{L} \models$ (ex2) (ex2)
- $\mathbf{L} \models \overline{\overline{w}}x(z+y) \lessapprox w + xz + xy$ (vi)
- $\mathbf{L} \models w + \overline{\overline{w}}x(z+y) \approx \overline{\overline{w}}(w + xz + xy)$ $(\mathrm{di}_4)$
- $\mathbf{L} \models z + \overline{\overline{z}}x(z+y) \approx \overline{\overline{z}}(z + xy)$ (di)
- $\mathbf{L} \models z + \overline{\overline{z}}x(z+y) \lessapprox z + xy$ (di≤)
- $\mathbf{L} \models z + \overline{\overline{w}}x(z+y) \lessapprox w + z + xy$

Proof reminds of the characterisation of each of the two distributive laws by a universal inequality

# Side note: some identities of $\mathcal{V}i\mathcal{L}$

**Lemma (MB): $\mathcal{V}i\mathcal{L}$ satisfy**

- $x(y+z)\overline{\overline{xz}} \lessapprox xy + xz$
- $x(y+z)\overline{\overline{xz}} \approx (xy+xz)\overline{\overline{xz}} \approx xy\overline{\overline{xz}} + xz$
- $x(\overline{x}+y)\overline{\overline{z}} \lessapprox xy + z$
- $x\overline{\overline{xy}}(\overline{x}+y) \approx x\overline{\overline{y}}(\overline{x}+y) \approx xy$
- $\overline{\overline{z}}(x+y) \lessapprox \overline{\overline{z}}x + \overline{\overline{z}}y + z$
- $\overline{\overline{x}}(x+y) \approx x + \overline{\overline{x}}y$
- $\overline{\overline{x}}(x+\overline{x}) \approx x$

# Characterisation of (cl)

**Characterisation (MB): For $\mathbf{L} \in \mathcal{FL}$ TFAE**

- $\mathbf{L} \models$ (ex3) (ex3)
- $\mathbf{L} \models \overline{w(xy+xz)}w \lessapprox x(y+z) + \overline{x(y+z)}$ (cl)
- $\mathbf{L} \models$
  - $\overline{xy+xz} \lessapprox x(y+z) + \overline{x(y+z)}$ (cl1)
    and
  - $\overline{wy}w \lessapprox y + \overline{y}$ (cl2)

- in (cl): $w = 1 \rightsquigarrow$ (cl1)
- in (cl): $x = 1$, $z = y \rightsquigarrow$ (cl2)
- $$\begin{aligned} \overline{w(xy+xz)}w &\overset{\text{(cl2)}}{\lessapprox} xy + xz + \overline{xy+xz} \\ &\overset{\text{(cl1)}}{\lessapprox} xy + xz + x(y+z) + \overline{x(y+z)} \\ &\approx x(y+z) + \overline{x(y+z)} \qquad \text{(cl)} \end{aligned}$$

# Combining the characterisations

**Theorem (MB): The variety $\mathcal{E}x\mathcal{L}$ can be axiomatised by…**

…extending the axioms of fundamental lattices by each of

- (ex) — 6 variables
- (ex1), (ex2), (ex3) — 6 variables
- (nu), (vi), (cl) — 4 variables
- (nu), (di), (cl1), (cl2) — 3 variables

# What Aguilera & Massas actually prove (without specifically not(ic)ing)

$\mathcal{OL} \vee \mathcal{HL} \subseteq \mathcal{ExL}$

$\mathcal{ExL} \subseteq \mathcal{OL} \vee \mathcal{HL}$

- $\mathcal{ExL} \subseteq \mathcal{NuL} \cap \mathcal{ViL} \cap \mathcal{ClL} = \mathcal{NuL} \cap \mathcal{ViL} \cap \mathcal{Cl1L} \cap \mathcal{Cl2L} \subseteq \mathcal{FL}$
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Thm (Aguilera & Massas + MB): $\mathcal{ExL} = \mathcal{OL} \vee \mathcal{HL}$ is… (4 vars!)

…the subvariety given by adding to $\mathcal{FL}$-axioms:(nu), (vi), (cl1), (cl2)

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Thm (Aguilera & Massas + MB): $\mathcal{ExL} = \mathcal{OL} \vee \mathcal{HL}$ is… (3 vars!)

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# Joining Aguilera & Massas with MB

The variety $\mathcal{E}x\mathcal{L} = \mathcal{OL} \vee \mathcal{HL}$ can be axiomatised by (3 vars!)

bounded lattice identities and

- $x\overline{x} \approx 0$ (semicomplementation)
- $\overline{x} \cdot \overline{xy} \approx \overline{x}$ (order reversal)
- $x\overline{\overline{x}} \approx x$ (extensionality)
- $\overline{\overline{x}} \cdot \overline{\overline{y}} \approx \overline{\overline{xy}}$ (nu=, ·-hom)
- $z + \overline{\overline{z}}x(z + y) \approx \overline{\overline{z}}(z + xy)$ (di)
- $\overline{xy + xz}\left(x(y + z) + \overline{x(y + z)}\right) \approx \overline{xy + xz}$ (cl1=)
- $\overline{xy}x(y + \overline{y}) \approx \overline{xy}x$ (cl2=)

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- $x \lessapprox \overline{\overline{x}}$ (extensionality)
- $x \cdot \overline{\overline{y}} \lessapprox \overline{\overline{xy}}$ (nu$\leq$, $\cdot$-hom)
- $z + \overline{\overline{z}}x(z+y) \lessapprox z + xy$ (di$\leq$)
- $\overline{xy + xz} \lessapprox x(y+z) + \overline{x(y+z)}$ (cl1)
- $\overline{xy}x \lessapprox y + \overline{y}$ (cl2)

# Separating the influence of distributivity

**We know**

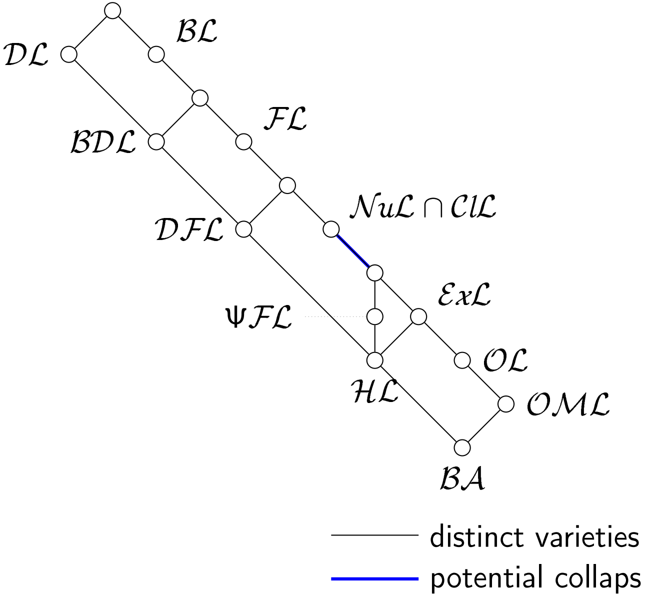
- $\mathcal{HL} \vee \mathcal{OL} = \mathcal{E}x\mathcal{L} = \mathcal{N}u\mathcal{L} \cap \mathcal{C}l\mathcal{L} \cap \mathcal{V}i\mathcal{L} \subseteq \mathcal{N}u\mathcal{L} \cap \mathcal{C}l\mathcal{L}$
- $\mathcal{HL} = \mathcal{DFL} \cap \Psi\mathcal{FL}$ (distr. & pseudocompl.)

**Observation (MB):**

- actually: $\Psi\mathcal{FL} \vee \mathcal{E}x\mathcal{L} \subseteq \mathcal{N}u\mathcal{L} \cap \mathcal{C}l\mathcal{L}$
  but: $\Psi\mathcal{FL} \nsubseteq \mathcal{V}i\mathcal{L}$

suggests that influence of $\mathcal{DFL}$ through $\mathcal{HL}$ on $\mathcal{E}x\mathcal{L}$ is only in terms of (vi) $\iff$ (di)

# Summarising lattice of $\langle \cdot, +, 0, 1, \bar{\ } \rangle$-varieties



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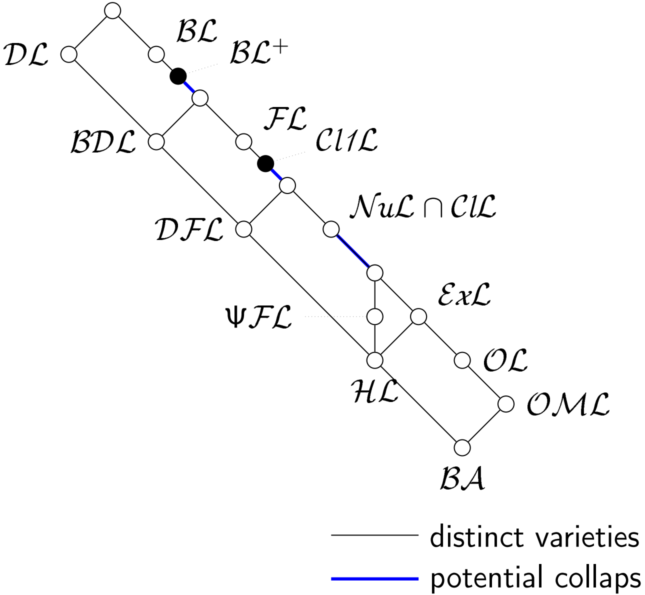

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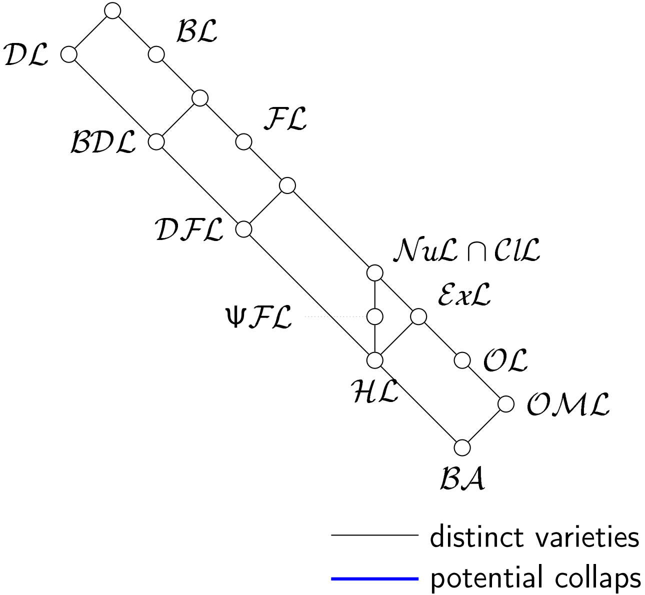

# The end

$x\overline{xy} \lessapprox y + \overline{y}$ (cl2)

If we have proof that I am grateful for your attention and we can refute that I provably am grateful for your attention and you provably were actually listening to what I was saying, then we have proof of the fact that you were actually listening, or we can refute that, and we know which one of the two is the case.