Ideals in universal algebra IV: Definability of principal ideals

Blansko, 8-12 September 2025

(Dual) Browuerian semilattices

A **Brouwerian semilattice** [6] is an algebra $\langle A, \to, \wedge, 1\rangle$ such that for any $a, b, c \in A$

1. $\langle A, \wedge, 1\rangle$ is an upper bounded semilattice;
2. $a \to a = 1$;
3. $(a \to b) \wedge a = (b \to a) \wedge b$;
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If **A** is a Brouwerian semilattice and $a, b, c \in A$, then

$$c \leq a \to b \qquad \text{if and only if} \qquad a \wedge c \leq b.$$

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The variety BS of Brouwerian semilattice is of course ideal determined. Moreover it can be shown that the class of $\rightarrow$-subreduct of BS coincides with the variety HI of Hilbert algebras and that the congruences (hence the ideals) of a Brouwerian semilattice coincide with those of its $\rightarrow$-reduct.

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A **dual Brouwerian semilattice** is a join semilattice with dual relative pseudocomplementation, i.e. an algebra $\langle A, \vee, *, 0\rangle$ such that

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The distinction between Brouwerian semilattices (Hilbert algebras) and dual Brouwerian semilattices (dual Hilbert algebras) is of course purely notational.

Definability of principal ideals

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If K is a class of algebras, we say that K has **definable principal ideals** (DPI) if there is a first order formula $\Psi(x, y, y_1, \ldots, y_n)$ in the language of K such that for all $\mathbf{A} \in \mathrm{K}$, $a, b \in A$

$$a \in (b)_{\mathbf{A}} \qquad \text{if and only if} \qquad \mathbf{A} \models \exists y_1, \ldots, y_n \, \Psi(a, b, y_1, \ldots, y_n).$$

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Proposition

[2] Let K *be a class of algebras with a constant* 0*.*

1. *If* K *has normal ideals and has definable principal congruences, then* **K** *has definable principal ideals.*
2. *If* K *is contained in an ideal determined variety and has definable principal ideals then* K *has definable principal congruences.*

We say that **K** has **equationally definable principal ideals in the broad sense** (EDPI$^{\#}$ for short) if there are terms p_i, q_i $i = 1, \dots, k$ such that for all $\mathbf{A} \in \mathsf{K}$, $a, b \in A$

$$a \in (b)_{\mathbf{A}} \text{ if and only if } \exists u_1, u_2, \dots \in A \text{ s.t.}$$
$$p_i(a, b, u_1, u_2, \dots) = q_i(a, b, u_1, u_2, \dots) \text{ for all } i = 1, \dots, k$$

A class K has a **uniform implicit term** p for principal ideals (UIT) if for any $\mathbf{A} \in \mathsf{K}$ $a, b \in A$

$$a \in (b)_{\mathbf{A}} \text{ if and only if } \exists u_1, u_2, \dots \in A \text{ s.t.} \qquad p(a, b, u_1, u_2, \dots) = 0.$$

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Next K has a **uniform explicit term** $q(x_1, \dots, x_n, y)$ for principal ideals (UET) if q is an ideal term in y and moreover, for any $\mathbf{A} \in \mathrm{K}$

$$a \in (b)_{\mathbf{A}} \text{ if and only if } \exists u_1, \dots, u_n \in A \text{ s.t.} \qquad q(u_1, \dots, u_n, b) = a.$$

A variety V has **factorable principal ideals on direct products** if, whenever $\mathbf{A}_i \in \mathrm{V}$ and $b \in \prod_{i \in I} \mathbf{A}_i$,

$$\prod_{i \in I} (b_i)_{\mathbf{A}_i} \subseteq (b)_{\mathbf{A}}.$$

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A variety V has a **test algebra for principal ideals**, if there exists an $\mathbf{A} \in V$ and $a, b \in A$, such that

- $a \in (b)_{\mathbf{A}}$;
- for any $\mathbf{B} \in V$ and $a', b' \in \mathbf{B}$, if $a' \in (b')_{\mathbf{B}}$ then there is a homomorphism φ of $\mathbf{A}$ into $\mathbf{B}$ such that $\varphi(a) = a'$ and $\varphi(b) = b'$.

Theorem

[2] For a subtractive variety V *the following are equivalent:*

1. V *has a UET.*
2. V *has a UIT.*
3. V *has EDPI*$^{\#}$.
4. V *has factorable principal ideals on direct products.*
5. V *has a test algebra for principal ideals.*

Assume (1) and let $q(x_1, \ldots, x_k, y)$ be a UET for K.

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$$p(x_1, \ldots, x_k, x, y) = s(x, q(x_1, \ldots, x_k, y)).$$

If $a \in (b)_{\mathbf{A}}$ then there are $u_1, \ldots, u_n \in A$ such that $q(u_1, \ldots, u_n, b) = a$.

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On the other hand if for some $u_1, \ldots, u_k \in A$, $p(u_1, \ldots, u_k, a, b) = 0$, then

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Since q is an ideal term in y and $q(\vec{u}, b) \in (b)_{\mathbf{A}}$ we conclude that $a \in (b)_{\mathbf{A}}$. This shows that p is a UIT for K.

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We consider a *subset* F of V such that for every finitely generated algebra $\mathbf{A} \in \mathrm{V}$ and for every $a, b \in A$ with $a \in (b)_{\mathbf{A}}$ there are: an algebra $\mathbf{A}' \in \mathrm{F}$, $a', b' \in A'$ with $a' \in (b')_{\mathbf{A}}$ and an isomorphism $\varphi : \mathbf{A} \longrightarrow \mathbf{A}'$ with $\varphi(a) = a'$ and $\varphi(b) = b'$.

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is a test algebra for principal ideals.

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Then, if $\mathbf{B} \in V$, $a', b' \in \mathbf{B}$ and $a' \in (b')_{\mathbf{B}}$, we get

$$a' = \varphi(a) = \varphi(q(u_1, \ldots, u_k, b)) = q(\varphi(u_1), \ldots, \varphi(u_k), b').$$

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Then, if $\mathbf{B} \in \mathrm{V}$, $a', b' \in \mathbf{B}$ and $a' \in (b')_{\mathbf{B}}$, we get

$$a' = \varphi(a) = \varphi(q(u_1, \ldots, u_k, b)) = q(\varphi(u_1), \ldots, \varphi(u_k), b').$$

Conversely if $a' = q(\varphi(u_1), \ldots, \varphi(u_k), b')$, being q an ideal term in y we get $a' \in (b')_{\mathbf{B}}$. So q is a UET for V and (1) holds.

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For subtractive varieties with EDPI we can get a strengthening of the previous theorem.

Theorem

[1] [2] For a subtractive variety V *the following are equivalent.*

1. V *has EDPI.*
2. *There are binary terms* p_i, $i = 1, \ldots, n$ *such that*

$$a \in (b)_{\mathbf{A}} \quad \text{if and only if} \quad p_i(a, b) = 0 \quad i = 1, \ldots, n.$$

3. *There is a binary term* $p(x, y)$ *such that*

$$a \in (b)_{\mathbf{A}} \quad \text{if and only if} \quad p(a, b) = 0 \quad i = 1, \ldots, n.$$

4. *For any family* $(\mathbf{A}_i : i \in I)$ *of algebras in* V *and for any subalgebra* $\mathbf{B}$ *of* $\prod_{i \in I} \mathbf{A}$ *for any* $a, b \in B$,

$$a \in (b)_{\mathbf{B}} \quad \text{if and only if} \quad a_i \in (b_i)_{\mathbf{A}_i}, \; i \in I.$$

5. *There exists an* $\mathbf{A} \in \mathsf{V}$ *generated by two elements* a *and* b*, such that*
   (i) $a \in (b)_{\mathbf{A}}$;
   (ii) *for any* $\mathbf{B} \in \mathsf{V}$ *and* $a', b' \in \mathbf{B}$*, if* $a' \in (b')_{\mathbf{B}}$ *then there is a homomorphism* φ *of* $\mathbf{A}$ *into* $\mathbf{B}$ *such that* $\varphi(a) = a'$ *and* $\varphi(b) = b'$.

Theorem

6. *There is a ternary term* $p(x, y, z)$ *such that* $p(x, y, 0) \approx 0$ *holds in* V *and for any algebra* $\mathbf{A} \in \mathrm{V}$, $a, b \in A$, $a \in (b)_{\mathbf{A}}$ *if and only if* $p(a, b, b) = a$.
7. *For any* $\mathbf{A} \in \mathrm{V}$ *the semilattice* $\mathrm{CI}(\mathbf{A})$ *is a dual Brouwerian semilattice.*

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The proofs of equivalences (1)-(6) go along the lines of those of the previous theorem. We will only show that (7) fits well.

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In fact let $I = 0/\theta$ for some $\theta \in \mathrm{Con}(\mathbf{A})$; then

$$\begin{aligned} a \in (b)_{\mathbf{A}} \vee I \text{ if and only if } & a \in (b)_{\mathbf{A}} \vee 0/\theta \\ \text{if and only if } & a/\theta \in (b/\theta)_{\mathbf{A}/\theta} \\ \text{if and only if } & p_i(a/\theta, b/\theta) = 0/\theta \text{ for } i = 1, \ldots, n \\ \text{if and only if } & p_i(a, b) \in I \text{ for } i = 1, \ldots, n. \end{aligned}$$

It follows that the operation

$$(a) * (b) = (p_1(a, b), \ldots, p_n(a, b))_{\mathbf{A}}$$

is a dual relative pseudocomplementation in $\mathrm{CI}(\mathbf{A})$ for any two principal ideals of $\mathbf{A}$.

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But it is a general fact (see [7], Lemma 4) that, if any two elements of a generating set of a join semilattice have a dual pseudocomplement, then the semilattice is dually Brouwerian.

For the converse assume that $\mathrm{CI}(\mathbf{A})$ is dually Brouwerian for any $\mathbf{A} \in \mathsf{V}$.

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Let $\mathbf{F}$ be the algebra freely generated in V by $\{x, y, v_j\}_{j \in \omega}$.

By hypothesis $(x)_{\mathbf{F}} * (y)_{\mathbf{F}}$ exists in $\mathrm{CI}(\mathbf{F})$, hence there are terms $r_i(x, y, v_1, v_2, \dots)$, $i = 1, \dots, n$ such that

$$(x)_{\mathbf{F}} * (y)_{\mathbf{F}} = \bigvee_{i=1}^{n} (r_i(x, y, v_1, v_2, \dots))_{\mathbf{F}}.$$

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Then $J = \varphi^{-1}(0) \in \mathrm{Id}(\mathbf{B})$ and we have

$$\begin{aligned} a \in (b)_{\mathbf{B}} & \text{ if and only if } \varphi(x) \in (\varphi(y))_{\mathbf{B}} \\ & \text{ if and only if for some } t \in (y)_{\mathbf{F}},\ (x, t) \in ker\varphi \\ & \text{ if and only if } x \in (y)_{\mathbf{F}} \vee J \\ & \text{ if and only if } r_i(x, y, v_1, v_2, \dots) \in J \ \text{ for } i = 1, \dots, n \\ & \text{ if and only if } \varphi(r_i(x, y, v_1, v_2, \dots)) = 0 \ \text{ for } i = 1, \dots, n \\ & \text{ if and only if } r_i(a, b, a, a, \dots) = 0 \ \text{ for } i = 1, \dots, n \\ & \text{ if and only if } p_i(a, b) = 0 \ \text{ for } i = 1, \dots, n. \end{aligned}$$

Corollary

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If $\mathbf{A} \in \mathsf{V}$ we already know that $\mathrm{Id}(\mathbf{A})$ is isomorphic with the ideal lattice of $\mathrm{CI}(\mathbf{A})$.

Corollary

Every subtractive variety V *with EDPI is ideal distributive.*

If $\mathbf{A} \in \mathsf{V}$ we already know that $\mathrm{Id}(\mathbf{A})$ is isomorphic with the ideal lattice of $\mathrm{CI}(\mathbf{A})$.

By the previous theorem the latter is a dual Brouwerian semilattice and it is well known that the ideal lattice of a dual Brouwerian semilattice is distributive.

Let us remark that if $\mathbf{A}$ is a Hilbert algebra (or a Brouwerian semilattice) $a, b \in \mathbf{A}$ and $*$ is the dual relative pseudocomplementation, then

$$a \in (b)_{\mathbf{A}} \qquad \text{if and only if} \qquad a * b = 0.$$

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This means that the binary term giving relative pseudocomplementation witnesses *both* subtractivity and EDPI. In other words in a Brouwerian semilattice $\mathbf{A}$

$$(a)_{\mathbf{A}} * (b)_{\mathbf{A}} = (a * b)_{\mathbf{A}}.$$

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Hence the set $\mathrm{PI}(\mathbf{A})$ of principal ideals of $\mathbf{A}$ is closed under $*$ and $\langle \mathrm{PI}(\mathbf{A}), *, (0)_{\mathbf{A}} \rangle$ is a dual Hilbert algebra.

Let us remark that if **A** is a Hilbert algebra (or a Brouwerian semilattice) $a, b \in \mathbf{A}$ and $*$ is the dual relative pseudocomplementation, then

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Really we can go even further, since we can show that any algebra in a subtractive variety with EDPI has a "weak structure" closely resembling a dual Hilbert algebra.

Theorem

[2] Let V *be subtractive and EDPI. Then there exists a binary term* $x * y$ *with the following properties.*

1. *For all* $\mathbf{A} \in \mathsf{V}$ *and* $a \in A$

$$\begin{aligned} a * a &= 0 \\ a * 0 &= 0 \\ 0 * a &= a \\ b \in (a)_{\mathbf{A}} & \text{ if and only if } a * b = 0. \end{aligned}$$

2. *The relation* $\leq$ *defined by* $a \leq b$ *if and only if* $b * a = 0$ *is reflexive and transitive. The associated equivalence relation* $\approx_{\mathbf{A}}$ *is a congruence of* $\mathbf{A}^* = \langle A, *, 0 \rangle$ *and* $\mathbf{A}^* / \approx_{\mathbf{A}}$ *is a dual Hilbert algebra isomorphic with* $\langle \mathrm{PI}(\mathbf{A}), *, (0)_{\mathbf{A}} \rangle$.
3. *Any principal ideal of* $\mathbf{A}$ *is the union of a principal ideal of* $\mathbf{A}^* / \approx_{\mathbf{A}}$ *and viceversa. In fact* $(a)_{\mathbf{A}} = \bigcup (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$.

Suppose that $s(x, y)$ is the witness of subtractivity.

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Then, since V has EDPI, from point (6) of the characterization theorem, we get the existence of a ternary term $p(x, y, z)$ such that, for any $\mathbf{A} \in \mathrm{V}$ and $a, b \in A$

$$p(b, a, 0) = 0 \qquad p(b, a, a) = b \text{ if and only if } b \in (a)_{\mathbf{A}}.$$

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$$p(b, a, 0) = 0 \qquad p(b, a, a) = b \text{ if and only if } b \in (a)_{\mathbf{A}}.$$

Define $x * y = s(y, p(y, x, x))$. Then

$$\begin{aligned} a * a &= s(a, p(a, a, a)) = s(a, a) = 0; \\ a * 0 &= s(0, p(0, a, a)) = s(0, 0) = 0; \\ 0 * a &= s(a, p(a, 0, 0)) = s(a, 0) = a. \end{aligned}$$

Next, if $b \in (a)_{\mathbf{A}}$, then

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Conversely, if $a * b = 0$, then $s(b, p(b, a, a)) = 0$. Since $0 \in (a)_{\mathbf{A}}$ and $p(b, a, a) \in (a)_{\mathbf{A}}$ ($p(x, y, z)$ is an ideal term in z), subtractivity yields $b \in (a)_{\mathbf{A}}$ as well. This takes care of (1).

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from $\mathbf{A}$ to $\mathrm{PI}(\mathbf{A})$.

Then $(a)_{\mathbf{A}} * (b)_{\mathbf{A}} = (a * b)_{\mathbf{A}}$, therefore the mapping is a homomorphism from $\mathbf{A}^*$ to $\langle \mathrm{PI}(\mathbf{A}), *, (0)_{\mathbf{A}} \rangle$, whose kernel coincides with $\approx_{\mathbf{A}}$. Hence (2) follows.

Finally if $b \in (a)_{\mathbf{A}}$ then $a * b = 0$. This implies

$$(a * b)/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}}$$

and so

$$a/\approx_{\mathbf{A}} * b/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}} .$$

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But $\mathbf{A}^*/\approx_{\mathbf{A}}$ is a dual Hilbert algebra, thus it has EDPI with witness term $*$.

This implies

$$b/\approx_{\mathbf{A}} \in (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$$

and so

$$b \in \bigcup (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}.$$

Next if $b \in \bigcup(a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$, then $b \in c/\approx_{\mathbf{A}} \in (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$, therefore $b \approx_{\mathbf{A}} c$ and

$$a/\approx_{\mathbf{A}} * c/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}} .$$

Next if $b \in \bigcup(a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$, then $b \in c/\approx_{\mathbf{A}} \in (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$, therefore $b \approx_{\mathbf{A}} c$ and

$$a/\approx_{\mathbf{A}} * c/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}} .$$

But this implies $(a * c)/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}}$ and so $a * c = 0$ (since $0/\approx_{\mathbf{A}} = \{0\}$, via (1)).

Next if $b \in \bigcup(a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$, then $b \in c/\approx_{\mathbf{A}} \in (a/\approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$, therefore $b \approx_{\mathbf{A}} c$ and

$$a/\approx_{\mathbf{A}} * c/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}} .$$

But this implies $(a * c)/\approx_{\mathbf{A}} = 0/\approx_{\mathbf{A}}$ and so $a * c = 0$ (since $0/\approx_{\mathbf{A}} = \{0\}$, via (1)).

From $a * c = 0$ and $c * b = 0$ we get (via (2)) $a * b = 0$ and therefore $b \in (a)_{\mathbf{A}}$.

As a matter of fact the previous result has a converse which we state without proof.

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Theorem

[2] Let V be a variety with a constant 0 and such that the following hold.

1. *There exists a binary term $x * y$ such that for any $\mathbf{A} \in \mathsf{V}$ and $a \in A$*
$$a * a = 0$$
$$a * 0 = 0$$
$$0 * a = 0 \Rightarrow a = 0.$$
2. *The relation $\approx_{\mathbf{A}}$ defined by $a \approx b$ if and only if $a * b = b * a = 0$ is a congruence of $\mathbf{A}^* = \langle A, *, 0 \rangle$ and $\mathbf{A}^* / \approx_{\mathbf{A}}$ has EDPI defined by $u/ \approx_{\mathbf{A}} \in (v/ \approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}$ if and only if $u/ \approx_{\mathbf{A}} * v/ \approx_{\mathbf{A}} = 0/ \approx_{\mathbf{A}}$.*
3. *For any $a \in A$*
$$(a)_{\mathbf{A}} = \bigcup (a/ \approx_{\mathbf{A}})_{\mathbf{A}^*/\approx_{\mathbf{A}}}.$$

Then V is subtractive and has EDPI: for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$
$$a \in (b)_{\mathbf{A}} \qquad \text{if and only if} \qquad a * b = 0.$$

Meet and join generator terms

A class K has an n-**system of principal ideal intersection terms** if there are binary terms $q_1, \dots, q_n$ such that for any $\mathbf{A} \in$ K and $a, b \in A$,

$$(a)_{\mathbf{A}} \cap (b)_{\mathbf{A}} = \bigvee_{i=1}^{n} (q_i(a, b))_{\mathbf{A}}.$$

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$$(a)_{\mathbf{A}} \cap (b)_{\mathbf{A}} = \bigvee_{i=1}^{n} (q_i(a,b))_{\mathbf{A}}.$$

Theorem

[2] For a subtractive variety V the following are equivalent.

1. *V has an n-system of principal ideal intersection terms.*
2. *V is ideal distributive and the compact ideals of any algebra in V are closed under intersections.*

Assume (1) and let $q_1, \ldots, q_n$ be an n-system of principal ideal intersection terms for V.

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Note that $q_i(x, y)$ is a commutator term in x, y by definition, so for any $\mathbf{A} \in \mathrm{V}$ and $a, b \in A$ $[a, b]_{\mathbf{A}} = (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}}$.

Assume (1) and let $q_1, \ldots, q_n$ be an n-system of principal ideal intersection terms for V.

Note that $q_i(x, y)$ is a commutator term in x, y by definition, so for any $\mathbf{A} \in$ V and $a, b \in A$ $[a, b]_{\mathbf{A}} = (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}}$.

Therefore the commutator is neutral and thus V is ideal distributive.

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Therefore the commutator is neutral and thus V is ideal distributive.

This fact and the principal ideal intersection terms yield

$$\bigvee_{j=1}^{m} (a_j)_{\mathbf{A}} \cap \bigvee_{l=1}^{k} (b_l)_{\mathbf{A}} = \bigvee_{j=1}^{m} \bigvee_{l=1}^{k} \bigvee_{i=1}^{n} (q_i(a_j, b_l))_{\mathbf{A}},$$

so (2) holds.

Assume now (2) and let $\mathbf{F}$ be the algebra in V freely generated by $x, y, v_1, v_2, \dots$.

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Since the compact ideal are closed under intersections we have that

$$(x)_{\mathbf{F}} \cap (y)_{\mathbf{F}} = \bigvee_{i=1}^{n} (t_i(x, y, v_{i_1}, \ldots, v_{i_k}))_{\mathbf{F}}.$$

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Define $q_i(x, y) = t_i(x, y, x, \dots, x)$ for $i = 1, \dots, n$.

Suppose that $\mathbf{A} \in \mathsf{V}$ is finitely generated and let $a, b \in A$.

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Then there is a homomorphism f of $\mathbf{F}$ onto $\mathbf{A}$ such that $f(x) = f(v_{i_j}) = a$ and $f(y) = b$.

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Now

$c \in (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}}$ if and only if $c \in (f(x))_{\mathbf{A}} \cap (f(y))_{\mathbf{A}}$
if and only if $c \in [f(x), f(y)]_{\mathbf{F}}$ if and only if $c \in f([x, y]_{\mathbf{F}})$

if and only if $c \in f((x)_{\mathbf{F}} \cap (y)_{\mathbf{F}})$ if and only if $c \in f(\bigvee_{i=1}^{n} (t_i(x, y, v_{i_1}, \dots, v_{i_k}))_{\mathbf{F}})$

if and only if there is an ideal term t such that

$$c = f(t(u_1, \dots, u_n, t_1(x, y, v_{1_1}, \dots, v_{1_k}), \dots, t_n(x, y, v_{n_1}, \dots, v_{n_k})))$$

if and only if $c = t(f(u_1), \dots, f(u_n), t_1(a, b, a, \dots, a), \dots, t_n(a, b, a, \dots, a))$
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$c \in (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}}$ if and only if $c \in (f(x))_{\mathbf{A}} \cap (f(y))_{\mathbf{A}}$
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if and only if there is an ideal term t such that

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if and only if $c = t(f(u_1), \ldots, f(u_n), t_1(a, b, a, \ldots, a), \ldots, t_n(a, b, a, \ldots, a))$
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if and only if $c \in \bigvee_{i=1}^{n} (q_i(a, b))_{\mathbf{A}}$.

So the conclusion holds if $\mathbf{A}$ is finitely generated.

However, if $c \in (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}}$ then there is a finitely generated subalgebra $\mathbf{B}$ of $\mathbf{A}$ such that $c \in (a)_{\mathbf{B}} \cap (b)_{\mathbf{B}}$.

However, if $c \in (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}}$ then there is a finitely generated subalgebra $\mathbf{B}$ of $\mathbf{A}$ such that $c \in (a)_{\mathbf{B}} \cap (b)_{\mathbf{B}}$.

Therefore the conclusion holds in general and $q_1, \ldots, q_n$ is an n-system of principal ideal intersection terms for V.

The case $n = 1$ in the definition of n-system of principal ideal intersection terms deserves a special name: the binary term witnessing that is called a **meet generator** for V and is denoted by $\sqcap$.

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The case $n = 1$ in the definition of n-system of principal ideal intersection terms deserves a special name: the binary term witnessing that is called a **meet generator** for V and is denoted by $\sqcap$.

Then, for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$

$$(a)_{\mathbf{A}} \cap (b)_{\mathbf{A}} = (a \sqcap b)_{\mathbf{A}}.$$

Just by looking at the proof of the previous theorem one sees that a subtractive variety has a meet generator term if and only if it is ideal distributive and the meet of two principal ideals is principal.

If a subtractive EDPI variety V has a meet generator term $\sqcap$, then the principal ideals are closed under both intersection and dual relative pseudocomplementation.

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It follows that, for any $\mathbf{A} \in$ V, $\langle \mathrm{PI}(\mathbf{A}), *, \cap, (0)_{\mathbf{A}} \rangle$ is a $*, \cap$-subreduct of a dual Brouwerian semilattice.

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It follows that, for any $\mathbf{A} \in \mathsf{V}$, $\langle \mathrm{PI}(\mathbf{A}), *, \cap, (0)_{\mathbf{A}} \rangle$ is a $*, \cap$-subreduct of a dual Brouwerian semilattice.

Moreover, via the meet generator term and distributivity of ideals, the compact ideals themselves are closed under intersection, hence $\langle \mathrm{CI}(\mathbf{A}), *, \vee, \cap, (0)_{\mathbf{A}} \rangle$ is a dual relatively pseudocomplemented lattice.

A **join generator** for a pointed variety V is a binary term $x \sqcup y$ such that for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$

$$(a)_{\mathbf{A}} \vee (b)_{\mathbf{A}} = (a \sqcup b)_{\mathbf{A}}.$$

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Proposition

[2] Let V be a pointed variety; then the following are equivalent.

1. *The join of two principal ideals is principal.*
2. *Every compact ideal is principal.*
3. *There are a binary term $\sqcup$ and two ternary terms r and t such that*

$$\begin{aligned} 0 \sqcup 0 &\approx 0 \\ r(x, y, 0) \approx t(x, y, 0) &\approx 0 \\ r(x, y, x \sqcup y) &\approx x \\ t(x, y, x \sqcup y) &\approx y. \end{aligned}$$

4. *V has a join generator term.*

A **join generator** for a pointed variety V is a binary term $x \sqcup y$ such that for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$

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4. *V has a join generator term.*

Note that subtractivity is not needed.

If a subtractive variety has EDPI and a join generator term we can obtain a stronger characterization theorem.

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Theorem

[2] Let V *a subtractive EDPI variety in which the join of two principal ideals is principal. Then there are binary terms* $*$ *and* $\sqcup$ *such that the following hold.*

1. *For all* $\mathbf{A} \in \mathrm{V}$ *and* $a, b, c \in A$

$$a * a = 0 \qquad (c * a) * ((c * b) * (c * (a \sqcup b))) = 0$$
$$a * 0 = 0 \qquad (a \sqcup b) * b = (a \sqcup b) * a = 0$$
$$0 * a = a$$
$$b \in (a)_{\mathbf{A}} \text{ if and only if } a * b = 0$$

2. *The relation* $\leq$ *defined by* $a \leq b$ *if and only if* $b * a = 0$ *is reflexive and transitive. The associated equivalence relation* $\approx_{\mathbf{A}}$ *is a congruence of* $\mathbf{A}^{\sqcup} = \langle A, *, \sqcup, 0 \rangle$ *and* $\mathbf{A}^{\sqcup} / \approx_{\mathbf{A}}$ *is a dual Brouwerian semilattice isomorphic with* $\langle \mathrm{PI}(\mathbf{A}), *, \vee, (0)_{\mathbf{A}} \rangle$.
3. *Any principal ideal of* $\mathbf{A}$ *is the union of a principal ideal of* $\mathbf{A}^{\sqcup} / \approx_{\mathbf{A}}$ *and viceversa. In fact* $(a)_{\mathbf{A}} = \bigcup (a / \approx_{\mathbf{A}})_{\mathbf{A}^{\sqcup} / \approx_{\mathbf{A}}}$.

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Proposition

[2] Suppose $\mathbb{V}$ *is a subtractive variety with EDPI and join generator term . Then* $\mathbb{V}$ *has also a meet generator term.*

The proof of the previous theorem is similar to others we have already seen. More interesting is the following:

Proposition

[2] Suppose V *is a subtractive variety with EDPI and join generator term . Then* V *has also a meet generator term.*

First we observe that the equation

$$(x * (x * y)) \vee (y * (y * x)) \approx x \wedge y$$

holds in any dual relatively pseudocomplemented lattice (see for instance [8]).

Let then $\sqcup$ be the join generator for V and define

$$x \sqcap y = (x * (x * y)) \sqcup (y * (y * x)).$$

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Now let $\mathbf{A} \in$ V and $a, b \in A$.

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$$x \sqcap y = (x * (x * y)) \sqcup (y * (y * x)).$$

Now let $\mathbf{A} \in \mathrm{V}$ and $a, b \in A$.

Since V has EDPI and a join generator term

$$(a \sqcap b)_{\mathbf{A}} = [(a)_{\mathbf{A}} * ((a)_{\mathbf{A}} * (b)_{\mathbf{A}})] \vee [(a)_{\mathbf{A}} * ((a)_{\mathbf{A}} * (b)_{\mathbf{A}})] = (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}},$$

Let then $\sqcup$ be the join generator for V and define

$$x \sqcap y = (x * (x * y)) \sqcup (y * (y * x)).$$

Now let $\mathbf{A} \in \mathrm{V}$ and $a, b \in A$.

Since V has EDPI and a join generator term

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where we have used the fact that the compact ideals form a dual relatively pseudocomplemented lattice.

Let then $\sqcup$ be the join generator for V and define

$$x \sqcap y = (x * (x * y)) \sqcup (y * (y * x)).$$

Now let $\mathbf{A} \in \mathrm{V}$ and $a, b \in A$.

Since V has EDPI and a join generator term

$$(a \sqcap b)_{\mathbf{A}} = [(a)_{\mathbf{A}} * ((a)_{\mathbf{A}} * (b)_{\mathbf{A}})] \vee [(a)_{\mathbf{A}} * ((a)_{\mathbf{A}} * (b)_{\mathbf{A}})] = (a)_{\mathbf{A}} \cap (b)_{\mathbf{A}},$$

where we have used the fact that the compact ideals form a dual relatively pseudocomplemented lattice.

Hence $x \sqcap y$ is a meet generator for V.

Theorem

[2] Let V a subtractive variety with EDPI in which the join and the meet of two principal ideals is principal. Then there are binary terms $*$, $\sqcup$ and $\sqcap$ such that the following hold.

1. *For all* $\mathbf{A} \in \mathrm{V}$ *and* $a, b, c \in A$

$$\begin{array}{ll} a * a = 0 & (c * a) * ((c * b) * (c * (a \sqcup b))) = 0 \\ a * 0 = 0 & (a \sqcup b) * b = (a \sqcup b) * a = 0 \\ 0 * a = a & (a * c) * ((b * c) * ((a \sqcap b) * c)) = 0 \\ b \in (a)_{\mathbf{A}} \text{ iff } a * b = 0 & a * (a \sqcap b) = b * (a \sqcap b) = 0. \end{array}$$

2. *The relation* $\leq$ *defined by* $a \leq b$ *iff* $b * a = 0$ *is reflexive and transitive. The associated equivalence relation* $\approx_{\mathbf{A}}$ *is a congruence of* $\mathbf{A}^{\sqcap} = \langle A, *, \sqcup, \sqcap, 0 \rangle$ *and* $\mathbf{A}^{\sqcap} / \approx_{\mathbf{A}}$ *is a relatively pseudocomplemented lattice isomorphic with* $\langle \mathrm{PI}(\mathbf{A}), *, \vee, \cap, (0)_{\mathbf{A}} \rangle$.
3. *Any principal ideal of* $\mathbf{A}$ *is the union of a principal ideal of* $\mathbf{A}^{\sqcap} / \approx_{\mathbf{A}}$ *and viceversa. In fact* $(a)_{\mathbf{A}} = \bigcup (a / \approx_{\mathbf{A}})_{\mathbf{A}^{\sqcap} / \approx_{\mathbf{A}}}$.

Pseudocomplemented semilattices

A **pseudocomplemented semilattice** is an algebra of type $\langle \wedge, *, 0 \rangle$ defined by the following identities

1. a set of identities defining meet semilattices;
2. $x \wedge (x \wedge y)^* = x \wedge y^*$;
3. $x \wedge 0^* = x$;
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Moreover if $\mathbf{L} \in \mathsf{PS}$ and $a \in L$, then a^* is the *pseudocomplement* of a, i.e. for any $b \in L$

$$b \leq a^* \qquad \text{if and only if} \qquad a \wedge b = 0.$$

It can be shown that

1. PS has EDPI
2. $x \sqcap y := x^{**} \wedge y^{**}$ is a meet generator term;
3. $x \sqcup y := (x^{*} \wedge y^{*})^{*}$ is a join generator term.
4. PS is not congruence regular so it is not ideal determined.

THANK YOU!

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