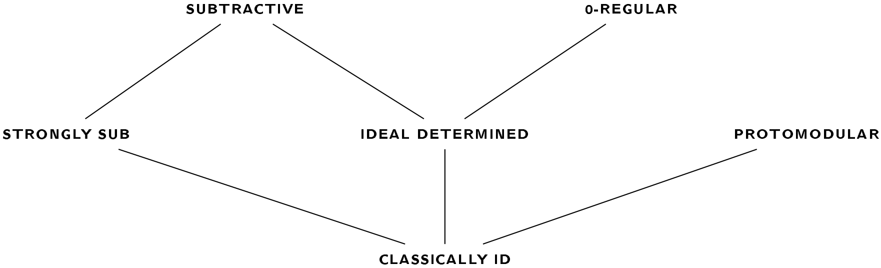
Ideals in universal algebra III: The Ideal Commutator

Blansko, 8-12 September 2025

The classes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A primer on the TC commutator

If α, β are congruences of any algebra **A**, then

1. $M(\alpha, \beta)$ is the set of all 2×2 matrices

$$\begin{pmatrix} t(\vec{a}^1, \vec{b}^1) & t(\vec{a}^1, \vec{b}^2) \\ t(\vec{a}^2, \vec{b}^1) & t(\vec{a}^2, \vec{b}^2) \end{pmatrix}$$

   where t is an $n+m$-ary term, $\vec{a}^1 \; \alpha \; \vec{a}^2$ (componentwise) and $\vec{b}^1 \; \beta \; \vec{b}^2$ (componentwise).

2. α centralizes β modulo γ (in symbols $C(\alpha, \beta; \gamma)$) if

$$\text{whenever } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(\alpha, \beta) \text{ and } a \; \gamma \; b \text{ then also } c \; \gamma \; d.$$

3. $[\alpha, \beta] = \bigwedge\{\gamma : C(\alpha, \beta; \gamma)\}$.

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2. the TC-commutator is symmetric, i.e. $[\alpha, \beta] = [\beta, \alpha]$;
3. $[\alpha, \beta]$ may be characterized in a different way (see [2], Chap. IV) and in particular the Hagemann-Herrmann definition is no more dependent on terms:

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3. $[\alpha, \beta]$ may be characterized in a different way (see [2], Chap. IV) and in particular the Hagemann-Herrmann definition is no more dependent on terms:
   let $\Delta_{\alpha,\beta}$ be the congruence on α (regarded as a subalgebra of $\mathbf{A} \times \mathbf{A}$), generated by all pairs $\langle\langle u, u\rangle, \langle v, v\rangle\rangle$ where $u \,\beta\, v$. Then $\langle a, b\rangle \in [\alpha, \beta]$ if and only if $\langle\langle a, b\rangle\langle b, b\rangle\rangle \in \Delta_{\alpha,\beta}$ if and only if for some c, $\langle\langle a, b\rangle\langle c, c\rangle\rangle \in \Delta_{\alpha,\beta}$.

Let V be any variety (with 0); $t(\vec{x}, \vec{y}.\vec{z})$ is a **commutator term** in $\vec{y}, \vec{z}$ if it is an ideal term in $\vec{y}$ and and ideal term in $\vec{z}$.

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For $\mathbf{A} \in$ V and nonempty $H, K \subseteq A$ we define the **commutator** of K and H as

$$[K, H] = \{t(\vec{a}, \vec{b}, \vec{c}) : t \text{ a commutator term in } \vec{y}, \vec{z},\ \vec{a} \in A, \vec{b} \in K, \vec{c} \in H\}$$

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We should have written $[K, H]_{\mathbf{A}}$ to stress the algebra or even $[K, H]_{\mathbf{A},\mathsf{V}}$ to stress the variety too. However we will see that at least the dependency from V can be avoided.

Proposition

[4] If V *is any variety,* $\mathbf{A} \in \mathsf{V}$ *and* $H, K \subseteq A$ *then:*

1. $[H, K]_{\mathbf{A},\mathsf{V}} \in \mathrm{Id}_{\mathsf{V}}(\mathbf{A})$;
2. $[H, K]_{\mathbf{A},\mathsf{V}} = [K, H]_{\mathbf{A},\mathsf{V}}$;
3. $[H, K]_{\mathbf{A},\mathsf{V}} = [(H)^{\mathsf{V}}_{\mathbf{A}}, (K)^{\mathsf{V}}_{\mathbf{A}}]_{\mathbf{A},\mathsf{V}}$.

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Really it is not hard to prove that the ideal commutator in subtractive varieties satisfies almost all the good properties of the TC-commutator in congruence modular varieties.

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Let V *be a subtractive variety,* $\mathbf{A} \in \mathsf{V}$ $I, K_\lambda \in \mathrm{Id}(\mathbf{A})$ *for* $\lambda \in \Lambda$. *Then*

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Let $\lambda, \mu \in \Lambda$ and let $a = t(\vec{a}, \vec{i}, \vec{l})$ where $t(\vec{x}, \vec{y}, \vec{z})$ is a commutator term in $\vec{y}, \vec{z}$, $\vec{a} \in A, \vec{i} \in I, \vec{l} \in K_\lambda \vee K_\mu$.

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Assume that $\vec{l} = h_1, \dots, h_r, m_1, \dots, m_t$ where $h_i \in K_\lambda$ and $m_j \in K_\mu$ and let $a' = t(\vec{a}, \vec{i}, 0, \dots, 0, m_1, \dots, m_t)$.

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Assume that $\vec{l} = h_1, \ldots, h_r, m_1, \ldots, m_t$ where $h_i \in K_\lambda$ and $m_j \in K_\mu$ and let $a' = t(\vec{a}, \vec{i}, 0, \ldots, 0, m_1, \ldots, m_t)$.

Then $a' \in [I, K_\mu]$ and moreover

$$s(t(\vec{x}, \vec{y}, z_1, \ldots, z_r, u_1, \ldots, u_k), t(\vec{x}, \vec{y}, \vec{0}, \vec{u}))$$

is a commutator term in $\vec{y}, \vec{z}$. Therefore $s(a, a') \in [I, K_\lambda]$, that yields $a \in [I, K_\lambda] \vee [I, K_\mu]$.

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Now a definition: if $\mathbf{A} \in \mathsf{V}$ and $I \in \mathrm{Id}(\mathbf{A})$ we define

$$I^{\#} = \mathrm{Sub}_{\mathbf{A}^2}(I \cup \{(a, a) : a \in J\}.$$

Then it is easy to show that $I \in \mathrm{Id}(\mathbf{A})$ if and only if $0/I^{\#} = I$.

Let now $\mathbf{A} \in \mathsf{V}$ be an algebra and $I, J \in \mathrm{Id}(\mathbf{A})$; we define

$$K_{I,J} = \text{the ideal of } I^{\#} \text{ generated by } \{(a,a) : a \in J\}$$
$$[I,J]_0 = 0/K_{I,J} = \{a : (0,a) \in K_{I,J}\}$$

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Proposition

For any $\mathbf{A} \in \mathrm{V}$ *and* $I, J \in \mathrm{Id}(\mathbf{A})$, $[I,J]_0$ *is an ideal and* $[I,J] \subseteq [I,J]_0$. *If* V *is s-subtractive then* $[I,J] = [I,J]_0$.

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Let $a = t(b, i, j) \in [I, J]$, where t is a commutator term in y, z and $b \in A$, $i \in I$, $j \in J$. Then in $I^{\#}$

$$(0, a) = t((b, b), (0, i), (j, j)) \in K_{I,J}.$$

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Then for some ideal term $t(x, y)$ in y and for some $(u, v) \in I^{\#}$ and $r \in J$ we have

$$(0, a) = t((u, v), (r, r))$$

i.e. $0 = t(u, r)$ and $a = t(v, r)$.

On the other hand, since $(u, v) \in I^{\#}$, there is a term $q(x, y)$, $h \in I$ and $b \in A$ with

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therefore $0 = t(q(0, b), r)$ and $a = t(q(h, b), r)$.

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Hence we get

$$\begin{aligned} a &= s(s(a,0),s(0,0)) \\ &= s(s(t(q(h,b),r),t(q(h,b),0)),s(t(q(0,b),r),t(q(0,b),0))) \end{aligned}$$

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But the term

$$s(s(t(q(y, x), z), t(q(y, x), 0)), s(t(q(0, x), z), t(q(0, x), 0)))$$

is a commutator term in y, z. Since $h \in I$ and $r \in J$ we get $a \in [I, J]$.

Now we can show that the commutators of two ideals in an algebra in a subtractive variety depends only on the algebra and not on the variety.

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Proposition

If $\mathcal{V}$ *is subtractive,* $\mathbf{A} \in \mathcal{V}$ *and* $I, J \in \mathrm{Id}(\mathbf{A})$*, then*

$$[I, J]_{\mathbf{A}} = \{t(\vec{a}, \vec{i}, \vec{j}) : t \text{ any term, } \vec{a} \in A, \vec{i} \in I, \vec{j} \in J \text{ and } t(\vec{a}, \vec{0}, \vec{0}) = t(\vec{a}, \vec{i}, \vec{0}) = t(\vec{a}, \vec{0}, \vec{j}) = 0\}$$

Sketch of the proof

Let

$$\Sigma_{I,J} = \mathrm{Sub}_{I^\# \times I^\#}(\{((0,0),(a,a)) : a \in J\};$$

and check that $K_{I,J} = 0/\Sigma_{I,J}$.

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Check also that, if $X, Y \subseteq A \times A$, then

$$\mathrm{Sub}_{\mathbf{A}\times\mathbf{A}}(X \cup \mathrm{Sub}_{\mathbf{A}\times\mathbf{A}}(Y)) = \mathrm{Sub}_{\mathbf{A}\times\mathbf{A}}(X \cup Y).$$

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Therefore $(c,d) \in K_{I,J}$ if and only if $\langle(0,0),(c,d)\rangle \in \Sigma_{I,J}$ if and only if there is a term $t(\vec{x},\vec{y},\vec{z})$ such that

$$\langle(0,0),(c,d)\rangle = t(\overrightarrow{\langle(0,0)(b,b)}\rangle, \overrightarrow{\langle(a,a),(a,a)}\rangle, \overrightarrow{\langle(0,i),(0,i)}\rangle)$$

for some $\vec{b} \in J$, $\vec{a} \in A$ and $\vec{i} \in I$.

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for some $\vec{b} \in J$, $\vec{a} \in A$ and $\vec{i} \in I$.

The conclusion follows.

From the previous Proposition we can infer other similar characterizations for $[I, J]$ in a subtractive algebra.

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2. $[I, J]_{\mathbf{A}} = \{s(s(t(\vec{i},\vec{j}), t(\vec{i},\vec{0})), s(t(\vec{0},\vec{j}), t(\vec{0},\vec{0}))) :$ *t a polynomial of* $\mathbf{A}, \vec{i} \in I, \vec{j} \in J\}$.

An easy observation

Lemma

Let $\mathbf{A}$, $\mathbf{B}$ *belong to a subtractive variety* V*; let* $I, J \in \mathrm{Id}(\mathbf{A})$ *and let* g *be a homomorphism from* $\mathbf{A}$ *onto* $\mathbf{B}$*. Then* $g([I, J]_{\mathbf{A}}) = [g(I), g(J)]_{\mathbf{B}}$.

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Let $u \in g([I, J]_{\mathbf{A}})$; then there is a commutator term for V in $\vec{y}, \vec{z}$ and elements $\vec{a} \in A$, $\vec{b} \in I$ and $\vec{c} \in J$ with

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The reverse inclusion is equally obvious.

An easy observation

Lemma

Let $\mathbf{A}$, $\mathbf{B}$ *belong to a subtractive variety* V*; let* $I, J \in \mathrm{Id}(\mathbf{A})$ *and let* g *be a homomorphism from* $\mathbf{A}$ *onto* $\mathbf{B}$*. Then* $g([I,J]_{\mathbf{A}}) = [g(I), g(J)]_{\mathbf{B}}$*.*

Let $u \in g([I,J]_{\mathbf{A}})$; then there is a commutator term for V in $\vec{y}, \vec{z}$ and elements $\vec{a} \in A$, $\vec{b} \in I$ and $\vec{c} \in J$ with

$$u = g(t(\vec{a}, \vec{b}, \vec{c})) = t(g(\vec{a}), g(\vec{b}), g(\vec{c})) \in [g(I), g(J)]_{\mathbf{B}}.$$

The reverse inclusion is equally obvious.

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Those, in our language, are *pure* (i.e. without parameters) commutator terms. Namely if $\mathbf{G}$ is a group and $\mathbf{N}, \mathbf{M} \lhd \mathbf{G}$ then

$$[\mathbf{N}, \mathbf{M}]_{\mathbf{G}} = Sub_{\mathbf{G}}(\{n^{-1}m^{-1}nm : n \in N, m \in M\}).$$

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In other words the only commutator term we have to concern about is $y^{-1}z^{-1}yz$ and this clearly implies that the commutator of $\mathbf{N},\mathbf{M}$ is the same in any group that contains both of them.

Commutator identities

Consider an algebraic language having symbols for the join, intersection, 0,1 and the commutator; identities in that language are called *commutator identities*. Note that $\mathrm{Id}(\mathbf{A})$ can be seen as a model of the language.

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We say that a class K of algebras **satisfies the commutator identity** $p \approx q$ and we will write

$$\mathsf{K} \models_{id} p \approx q,$$

if $p \approx q$ holds in $\mathrm{Id}(\mathbf{A})$ for all $\mathbf{A} \in \mathsf{K}$.

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Proposition

For any algebra $\mathbf{A}$ *the following are equivalent:*

1. $\mathbf{A} \models_{id} [x, y] = x \cap y \cap [A, A]$;
2. $\mathbf{A} \models_{id} [x, y \cap z] = [x, y] \cap z$;
3. $\mathbf{A} \models_{id} [x, y] = [x, A] \cap y$;
4. $\mathbf{A} \models_{id} [x, x] = x \cap [A, A]$;
5. $\mathbf{A} \models_{id} x \subseteq [A, A] \implies x = [x, x]$;
6. *for all* $a \in A$, *if* $a \in [A, A]$ *then* $[a, a] = (a)_{\mathbf{A}}$.

Is there an equivalent algebraic condition corresponding to the satisfaction of any of the conditions in the previous proposition? Yes, but we need some definitions.

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An algebra **A** is **ideal abelian** if $[A, A] = \{0\}$.

An algebra **A** is **ideal prime** if for all $I, J \in \mathrm{Id}(\mathbf{A})$, $[I, J] = \{0\}$ implies $I = \{0\}$ or $J = \{0\}$.

Theorem

[1] For a subtractive variety V *the following are equivalent:*

1. $\mathsf{V} \models_{id} [x, y \cap z] \approx [x, y] \cap z$;
2. *every ideal irreducible algebra in* V *is either ideal abelian or ideal prime.*

Assume (1) and let $I, J \in \mathrm{Id}(\mathbf{A})$ with $[I, J] = \{0\}$. Then

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Assume (2); by the previous proposition it is enough to show that if $\mathbf{A} \in \mathsf{V}$, $I \in \mathrm{Id}(\mathbf{A})$ and $I \subseteq [A, A]$ then $[I, I] = I$.

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Assume by contradiction that there exists $a \in I \setminus [I, I]$; using Zorn Lemma let U be maximal in

$$\{J \in \mathrm{Id}(\mathbf{A}) : [I, I] \subseteq J, a \notin J\}$$

and let $\theta \in \mathrm{Con}(\mathbf{A})$ such that $U = 0/\theta$.

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Let L be a nonzero ideal of $\mathbf{A}/\theta$; for some $J \supsetneq U$ we have $L = \{b/\theta : b \in J\}$ and for some $b \in J$, $(0, b) \notin \theta$, i.e. $b \in J - U$. So $a \in J$, namely $a/\theta \in L$ and $\mathbf{A}/\theta$ is ideal irreducible; by hypothesis $\mathbf{A}/\theta$ is either ideal abelian or ideal prime.

Observe that $[I, I] \subseteq U$, $I \not\subseteq U$ and

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therefore

$$[(U \vee I)/\theta, (U \vee I)/\theta]_{\mathbf{A}/\theta} = [U \vee I, U \vee I]/\theta \subseteq U/\theta = \{0/\theta\},$$

while $(U \vee I)/\theta \neq \{0/\theta\}$, since $a/\theta \in (U \vee I)/\theta$.

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Hence $\mathbf{A}/\theta$ is not ideal prime and so it must be ideal Abelian. This implies

$$\{0/\theta\} = [A/\theta, A/\theta]_{\mathbf{A}/\theta} = [A, A]/\theta$$

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It follows by contradiction that (2) implies (1).

Ideal neutral algebras

An algebra $\mathbf{A}$ is **ideal neutral** if the commutator of ideals reduces to the intersection.

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Proposition

For a subtractive variety V *the following are equivalent:*

1. V *is ideal distributive;*
2. *for all* $\mathbf{A} \in \mathsf{V}$ *and* $\theta, \varphi, \psi \in \mathrm{Con}(\mathbf{A})$

$$0/(\theta \vee \varphi) \wedge \psi = 0/(\theta \wedge \psi) \vee (\varphi \wedge \psi).$$

Theorem

[1] For a subtractive variety $\mathcal{V}$ *the following are equivalent:*

1. $\mathcal{V} \models_{id} [x, y] = x \cap y;$
2. $\mathcal{V}$ *is ideal distributive ;*
3. *there are four ternary terms* $q_1, \ldots, q_4$ *such that the following identities hold in* $\mathcal{V}$*:*

$$\begin{aligned} q_i(x, y, 0) &= 0 \qquad i = 1, \ldots, 4 \\ q_1(x, y, x) &= q_2(x, y, y) \\ q_3(x, y, x) &= q_4(x, y, s(x, y)) = s(x, q_1(x, y, x)); \end{aligned}$$

4. *there is a binary term* $b(x, y)$ *such that the following identities hold in* $\mathcal{V}$*:*

$$b(x, x) = 0 \qquad b(0, x) = 0 \qquad b(x, 0) = x.$$

Ideal abelian algebras

Let **A** be any algebra; **A** is called **abelian** (see [3]) if for every term $t(x, \vec{y})$, for every $a, b, \vec{u}, \vec{v} \in A$, if $t(a, \vec{u}) = t(a, \vec{v})$ then $t(b, \vec{u}) = t(b, \vec{v})$.

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By Mal'cev criterion, this is equivalent to saying that the diagonal of $D(\mathbf{A}) = \{(a, a) : a \in A\}$ is a congruence class of $\mathbf{A} \times \mathbf{A}$.

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$$\forall\, t(x, \vec{y}) \text{ term}, \ \forall\, u, v, \vec{a}, \vec{b} \in A \tag{TC$_i$}$$

$$s(t(u, \vec{a}), t(u, \vec{b})) = 0 \qquad \text{if and only if} \qquad s(t(v, \vec{a}), t(v, \vec{b})) = 0$$

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$$\forall\, t(x,\vec{y}) \text{ term}, \ \forall\, v, \vec{a}, \vec{b} \in A, \tag{TC$_0$}$$
$$s(t(0,\vec{a}), t(0,\vec{b})) = 0 \quad \text{if and only if} \quad s(t(v,\vec{a}), t(v,\vec{b})) = 0$$

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Lemma

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We need only observe that if $g : \mathbf{A} \longrightarrow \mathbf{B}$ is a onto homomorphism and $U, V \in \mathrm{Id}(\mathbf{B})$, then $g^{-1}(U), g^{-1}(V) \in \mathrm{Id}(\mathbf{A})$. Then we apply the lemma.

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For every subtractive variety V, $\mathsf{IAB}(\mathsf{V})$ *is a variety.*

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A more interesting observation is the following:

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[4] If V *is subtractive then* $\mathsf{IAB}(\mathsf{V})$ *is strongly subtractive.*

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Proposition

[4] If V *is subtractive then* $\mathsf{IAB}(\mathsf{V})$ *is strongly subtractive.*

We will show that if $\mathbf{A} \in \mathsf{IAB}(\mathsf{V})$ and $I \in \mathrm{Id}(\mathbf{A})$, then I^* is a subalgebra of $\mathbf{A} \times \mathbf{A}$.

First observe that if $t(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$, then the identity

$$s(t(\vec{x}, \vec{y}), t(\vec{z}, \vec{y})) \approx 0$$

holds in IAB(V), simply because the shown term is a commutator term in $\vec{x} * \vec{z}, \vec{y}$.

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Let f be an n-ary operation; then

$$s(f(u(x_1, y_1, 0), \ldots, u(x_n, y_n, 0)), f(\vec{y}))$$

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Therefore in IAB(V)

$$\begin{aligned} 0 &\approx s(s(f(u(x_1, y_1, 0), \dots, u(x_n, y_n, 0)), f(\vec{y})), s(f(u(y_1, y_1, 0), \dots, u(y_n, y_n, 0)), f(\vec{y}))) \\ &\approx s(s(f(u(x_1, y_1, 0), \dots, u(x_n, y_n, 0)), f(\vec{y})), s(f(\vec{y}), f(\vec{y}))) \\ &\approx s(f(u(x_1, y_1, 0), \dots, u(x_n, y_n, 0)), f(\vec{y})). \end{aligned}$$

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$$\begin{aligned} 0 &\approx s(s(f(u(x_1, y_1, 0), \ldots, u(x_n, y_n, 0)), f(\vec{y})), s(f(u(y_1, y_1, 0), \ldots, u(y_n, y_n, 0)), f(\vec{y}))) \\ &\approx s(s(f(u(x_1, y_1, 0), \ldots, u(x_n, y_n, 0)), f(\vec{y})), s(f(\vec{y}), f(\vec{y}))) \\ &\approx s(f(u(x_1, y_1, 0), \ldots, u(x_n, y_n, 0)), f(\vec{y})). \end{aligned}$$

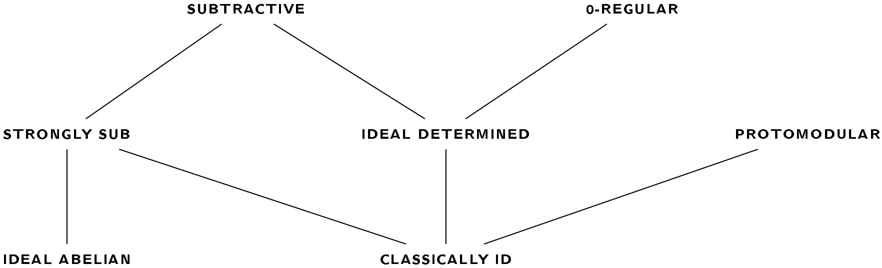
This means that

$$s(f(u(x_1, y_1, z_1), \ldots, u(x_n, y_n, z_n)), f(\vec{y}))$$

is an ideal term for IAB(V) in $\vec{z}$.

Therefore, if $(a_i, b_i) \in I^*$ then also $(f(\vec{a}), f(\vec{b})) \in I^*$. This proves the conclusion.

The classes (improved)



The three groups theorem

The last thing we show is a version of the so-called “Three groups theorem” for ideal abelian algebras.

The three groups theorem

The last thing we show is a version of the so-called "Three groups theorem" for ideal abelian algebras.

Proposition

[1] Let $\mathbf{A}$ *be subtractive. If* $\mathbf{M}_3$ *is a* 0-1-*sublattice of* $\mathrm{Id}(\mathbf{A})$, *then* $\mathbf{A}$ *is ideal Abelian. Moreover the following are equivalent:*

1. $\mathbf{A}$ *is ideal Abelian and non trivial;*
2. $\mathrm{Id}(\mathbf{A} \times \mathbf{A})$ *has* $\mathbf{M}_3$ *as a* 0-1-*sublattice;*
3. $\pi_1^{-1}(0)$ *and* $\pi_2^{-1}(0)$ *have a common complement in* $\mathrm{Id}(\mathbf{A} \times \mathbf{A})$*;*
4. *for some subdirect product* $\mathbf{S}$ *of* $\mathbf{A} \times \mathbf{A}$, $\mathrm{Id}(\mathbf{S})$ *has an* $\mathbf{M}_3$ *as a* 0-1-*sublattice.*

THANK YOU!

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