Ideals in universal algebra II: Regularity of congruences

Blansko, 8-12 September 2025

Ideal determined varieties

A variety V is congruence 0-regular if for all $\mathbf{A} \in$ V and $\theta, \varphi \in$ Con($\mathbf{A}$),$0/\theta = 0/\varphi$ implies $\theta = \varphi$. 0-regularity was introduced and characterized in [3]; when the variety is also subtractive, then it is said to be **ideal determined**.

Theorem

[5] For a variety V *the following are equivalent:*

1. V *is ideal determined;*
2. *any algebra in* V *has* 0*-regular and* 0*-permutable congruences;*
3. *there exists a natural number* m*, binary terms* $d_1(x,y),\ldots,d_m(x,y)$ *and a* $m+3$*-term* q *such that*

$$\begin{aligned}
&d_i(x,y)\approx 0 \quad \textit{for} \quad i=1,\ldots,m \ \textit{implies}\ x\approx y\\
&d_i(x,x)\approx 0 \quad \textit{for} \quad i=1,\ldots,m\\
&q(x,y,0,0,\ldots,0)\approx 0\\
&q(x,y,y,d_1(x,y),\ldots,d_m(x,y))\approx x
\end{aligned}$$

   hold in V*;*

4. *the mapping from* $\mathrm{Con}(\mathbf{A})\longrightarrow \mathrm{Id}(\mathbf{A})$ *defined by* $\theta\longmapsto 0/\theta$ *is a lattice isomorphism.*

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In an ideal determined variety the congruence permute at 0 and they are completely determined by the ideals. This does not mean however that the congruence must permute *away* from zero.

Implication algebras

An **implication algebra** (a.k.a. Hilbert algebra) is a $\to, 1$-subreduct of a Heyting algebra.

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&x \to x \approx 1\\
&(x \to y) \to x \approx x\\
&(x \to y) \to y \approx (y \to x) \to x\\
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Next if $x \to y \approx y \to x \approx 1$ then

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\begin{aligned}
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\end{aligned}
$$

which of course implies 1-regularity of congruences.

So the variety of implication algebras is ideal determined; it is not congruence permutable though as shown in [6]. In the same paper it is shown that it is congruence 3-permutable; this means that for any implication algebra **A** and $\theta \varphi \in \mathrm{Con}(\mathbf{A})$, $\theta \circ \varphi \circ \theta = \varphi \circ \theta \circ \varphi$.

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However this is not true in general; in [7] Rafetry proved that the variety of *lower BCK-semilattices* is ideal determined and 4-permutable but not 3-permutable.

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As a final fact in [4] Barbour and Raftery showed that for every n there is an ideal determined variety that is congruence n-permutable but not congruence $n+1$-permutable.

Bases for ideal terms

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This concept can be formalized as follows: if V is a variety a **base for the** V-**ideal terms** is any set T of ideal terms such that Tcontains 0, T is closed under compositions and the following holds: for any $\mathbf{A} \in \mathrm{V}$ and any $I \subseteq A$, $I \in \mathrm{Id}(\mathbf{A})$ if an only if for any $t(\vec{x}, \vec{y}) \in T$, $\vec{a} \in A$ and $\vec{b} \in I$, $t(\vec{a}, \vec{b}) \in I$.

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The interesting case is the one in which the base is finite and the reader can check in a minute that groups and rings have a finite base for their ideal terms. This is not a coincidence and to see why we first need a technical lemma.

Lemma

[3] For a variety V *the following are equivalent:*

1. *there is an* m *and binary terms* $d_1, \dots, d_m$ *such that the equivalence*

$$d_1(x, y) \approx \dots \approx d_m(x, y) \approx 0 \qquad \text{if and only if} \qquad x \approx y$$

hold in V*;*

2. *there is an* m*, binary terms* $d_1, \dots, d_m$ *an quaternary terms* $g_1, \dots, g_m$ *such that the equations*

$$g_1(x, y, d_1(x, y), 0) \approx x$$
$$g_i(x, y, 0, d_i(x, y)) \approx g_{i+1}(x, y, d_{i+1}(x, y), 0) \quad i = 1, \dots, m-1$$
$$g_m(x, y, 0, d_m(x, y)) \approx y$$

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3. V *is congruence* 0*-regular.*

Theorem

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Let $d_1, \ldots, d_m$ be the terms whose existence is guaranteed by the previous lemma. We first observe that if $\mathbf{A} \in \mathrm{V}$, $\theta \in \mathrm{Con}(\mathbf{A})$ and $I = 0/\theta \in \mathrm{Id}(\mathbf{A})$ then for all $a, b \in A$

$$(a, b) \in \theta \qquad \text{if and only if} \qquad d_i(a, b) \in I \quad i = 1, \ldots, m.$$

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$$(a, b) \in \theta \qquad \text{if and only if} \qquad d_i(a, b) \in I \quad i = 1, \ldots, m.$$

Next if f is an n-ary basic operation of $\mathcal{V}$ we consider the free algebra in $\mathcal{V}$ generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ and the ideal I generated by $\{d_i(x_k, y_k) : i = 1, \ldots, m,\ k = 1. \ldots, n\}$;

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$$(a, b) \in \theta \qquad \text{if and only if} \qquad d_i(a, b) \in I \quad i = 1, \ldots, m.$$

Next if f is an n-ary basic operation of V we consider the free algebra in V generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ and the ideal I generated by $\{d_i(x_k, y_k) : i = 1, \ldots, m,\ k = 1. \ldots, n\}$;

clearly $d_i(f(\vec{x}), f(\vec{y})) \in I$ for $i = 1, \ldots, m$ and thus there exist ideal terms $r_{i,f}$, $i = 1, \ldots, n$ such that

$$r_{i,f}(x, y, 0, \ldots, 0) \approx 0$$
$$r_{i,f}(x, y, d_1(x_1, y_1), \ldots, d_m(x_1, y_1), \ldots, d_1(x_n, y_n), \ldots, d_m(x_n, y_n)) \approx d_i(f(\vec{x}), f(\vec{y}))$$

hold in V.

Next since congruences are symmetric and transitive relations this means that for $i = 1, \ldots, n$ $d_i(x, y), d_i(z, y) \in I$ implies $d_i(x, z) \in I$. Hence there are terms q_i, $i = 1, \ldots, n$ such that

$$q_i(x, y, z, 0, \ldots, 0) \approx 0$$
$$q_i(x, y, z, d_1(x, y), \ldots, d_m(x, y), d_1(z, y), \ldots, d_m(z, y)) \approx d_i(x, z)$$

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Finally let q be the term whose existence is requested in point (4) of Theorem 1. We claim that

$$T = \{0, d_i, r_{i,f}, q_i, q\}$$

is a base for ideal terms for V.

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We have to check that T is closed under composition and that is is in fact a base. Both proofs are routine and are left as an exercise.

Protomodular varieties

In many ideal determined varieties there is a *strong additive structure* in the following sense: if $s(x, y)$ is the subtraction term, then there is another binary term $t(x, y)$ such that $t(y, s(x, y)) \approx y$ holds in the variety. This happens for instance in groups, rings and Boolean algebras.

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This property, when properly generalized, corresponds to an interesting categorical property called **protomodularity**. Let us stress that protomodularity is a concept defined in category theory; besides the rather unfortunate choice of the name when one tries to translate it into the universal algebraic language some adjustments must be made.

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Let V be a variety of algebras; if $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathrm{V}$ and $f : \mathbf{A} \longrightarrow \mathbf{C}$, $g : \mathbf{B} \longrightarrow \mathbf{C}$ are homomorphisms, the **pullback of A and B along C**, denoted by $\mathbf{A} \times_{\mathbf{C}} \mathbf{B}$ is the subalgebra of $\mathbf{A} \times \mathbf{B}$ consisting of all the pairs (a, b) such that $f(a) = g(b)$.

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It is readily checked that $\mathbf{A} \times_{\mathbf{C}} \mathbf{B}$ is a subalgebra of $\mathbf{A} \times \mathbf{B}$.

The Square Lemma

if $p_{\mathbf{A}}$, $p_{\mathbf{B}}$ are the projections of $\mathbf{A} \times_{\mathbf{C}} \mathbf{B}$ into $\mathbf{A}$, $\mathbf{B}$ then the square in Figure 1 has the **universal mapping property** in the following sense.

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Lemma

Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathrm{V}$, *consider the pullback of* $\mathbf{A}$ *and* $\mathbf{B}$ *along* $\mathbf{C}$, *let* $\mathbf{D} \in \mathrm{V}$ *such that* $f' : \mathbf{D} \longrightarrow \mathbf{A}$, $g' : \longrightarrow \mathbf{B}$ *be homomorphism. If* $ff' = gg'$, *then the function* $h : \mathbf{D} \longrightarrow \mathbf{A} \times_{\mathbf{C}} \mathbf{B}$ *defined by* $h(d) = (f'(d), g'(d))$ *is the unique homomorphism such that the following diagram commutes:*

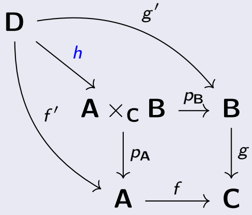

Figure: Pullback

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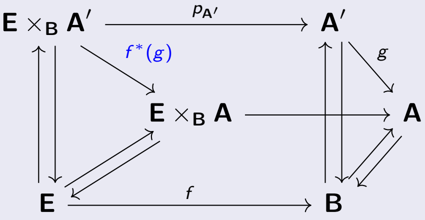
Theorem

Let $\mathbf{E}, \mathbf{B} \in \mathrm{V}$ and let $f : \mathbf{E} \longrightarrow \mathbf{B}$ be a homomorphism; if $\mathbf{A}, \mathbf{A}' \in r(\mathbf{B})$ and $g : \mathbf{A}' \longrightarrow \mathbf{A}$ is a homomorphism, then there is a unique homomorphism $f^*(g) : \mathbf{E} \times_{\mathbf{B}} \mathbf{A}' \longrightarrow \mathbf{E} \times_{\mathbf{B}} \mathbf{A}$ that makes the diagram in Figure 2 commute.



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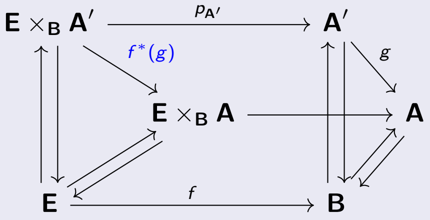

Figure

It is enough to apply the Square Lemma to the pullback of $\mathbf{E}$ and $\mathbf{A}$ along $\mathbf{B}$.

Let $\mathcal{R}(\mathbf{B})$ be the category whose objects are in $r(\mathbf{B})$ and whose morphisms are just homomorphisms between algebras in $r(\mathbf{B})$; then f^* can be seen as a functor from $\mathcal{R}(\mathbf{B})$ to $\mathcal{R}(\mathbf{E})$, where $f^*(\mathbf{A}) = \mathbf{E} \times_{\mathbf{B}} \mathbf{A}$.

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A variety $\mathbf{V}$ of algebras is **protomodular** if for all $\mathbf{E}, \mathbf{B} \in \mathsf{V}$ and for all $f : \mathbf{E} \longrightarrow \mathbf{B}$ the functor f^* reflects isomorphisms.

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In other words if for any $\mathbf{A}, \mathbf{A}' \in r(\mathbf{A})$ and $g : \mathbf{A}' \longrightarrow \mathbf{A}$

$$f^*(g) \text{ is an isomorphism} \qquad \text{implies} \qquad g \text{ is an isomorphism.}$$

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1. V *is protomodular;*
2. *if* $\mathbf{E} \leq \mathbf{B} \leq \mathbf{A}' \leq \mathbf{A} \in \mathsf{V}$ *with* $\mathbf{B}$ *a retract of* $\mathbf{A}$, *witness* α, *if* $\alpha^{-1}(\mathbf{E}) \leq \mathbf{A}'$, *then* $\mathbf{A}' = \mathbf{A}$;
3. *if* $\mathbf{E} \leq \mathbf{B} \leq \mathbf{A} \in \mathsf{V}$ *with* $\mathbf{B}$ *a retract of* $\mathbf{A}$, *witness* α, *then* $\mathbf{A} = \mathrm{Sub}_{\mathbf{A}}(\alpha^{-1}(E) \cup B)$.

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However since any variety V can be seen as a concrete category with free objects, the initial object, if it exists, is exactly the free algebra over the empty set.

Now for any variety $\mathbf{F}_{\mathsf{V}}(\emptyset)$ exists if and only if the language of V contains at least a constant and in this case it is the algebra generated by the constant elements. We will see in the next section that in this case protomodularity has a nice algebraic description.

Theorem

For a variety V *with a constant* 0 *the following are equivalent:*

1. V *is protomodular;*
2. *for all* $\mathbf{A}, \mathbf{B} \in \mathsf{V}$, *where* $\mathbf{B}$ *is a retract of* $\mathbf{A}$ *via* α *and* $\mathbf{E}$ *is the subalgebra of* $\mathbf{B}$ *generated by* 0, *then* $\mathbf{A} = \mathrm{Sub}_{\mathbf{A}}(\alpha^{-1}(E) \cup B)$;
3. *there is an* $n \in \mathbb{N}$, $e_1, \ldots, e_n \in E$, *an* $n+1$*-ary term* t *and binary terms* $d_1, \ldots, d_n$ *such that*

$$d_i(x, x) \approx e_i \qquad i = 1, \ldots, n$$
$$t(y, d_1(x, y), \ldots, d_n(x, y)) \approx x$$

holds in V.

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Since $x \in A$, there is an $n+1$-ary term t and binary terms $d_1, \ldots, d_n$ such that

$$x \approx t(y, d_1(x, y), \ldots, d_n(x, y))$$

where $t_1, \ldots, t_n \in E$. This means that $d_i(y, y) = \alpha(d_i(x, y))$ is in the subalgebra generated by the constants. It follows that there are $e_1, \ldots, e_n \in E_{\mathsf{V}}$ such that $d_i(x, x) \approx e_i$, $i = 1, \ldots, n$. This proves (3).

Assume now (3) and let $\mathbf{B} \leq \mathbf{A} \in \mathrm{V}$ where $\mathbf{B}$ is a retract of $\mathbf{A}$ via α. Then if $\mathbf{E}$ is the subalgebra of $\mathbf{B}$ generated by the constants and $a \in A$ we have

$$a = t(\alpha(a), d_1(\alpha(a), a), \ldots, d_n(\alpha(a), a))$$

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and $\alpha(d_i(\alpha(a), a)) = d_i(\alpha(a), \alpha(a)) = e_i \in E$. Therefore $\mathbf{A} = \mathrm{Sub}_{\mathbf{A}}(\alpha^{-1}(E) \cup B)$. Now if $\mathbf{E}' \leq \mathbf{B} \leq \mathbf{A} \in \mathsf{V}$, then $\mathbf{E} \leq \mathbf{E}'$ and, a fortiori, $\mathbf{A} = \mathrm{Sub}_{\mathbf{A}}(\alpha^{-1}(E') \cup B)$.

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Thus V is protomodular by the previous theorem.

We stress, even if there is no need, that we are not asking that the constants $e_1, \ldots, e_n$ be distinct.

Corollary

If $\mathbb{V}$ *is protomodular then it is congruence permutable. If the previous theorem holds for* $e_1 = \cdots = e_n = 0$, *then it is ideal determined.*

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If it is pointed then $E = \{0\}$ and so $d_i(x, x) \approx 0$ for $i = 1, \ldots, n$. Hence the term $s(x, y) = t(0, d_1(x, y), \ldots, d_n(x, y))$ is a subtraction term for V.

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Moreover if $d_i(x, y) \approx 0$ for $i = 1, \ldots, n$ then

$$\begin{aligned} x &\approx t(y, d_1(x, y), \ldots, d_n(x, y)) \approx t(y, 0, \ldots, 0) \\ &\approx t(y, d_1(y, y), \ldots, d_n(y, y)) \approx y. \end{aligned}$$

This shows that V is 0-regular and hence ideal determined.

Classically ideal determined varieties

Clearly if a variety is protomodular and **pointed**, i.e. there is exactly one constant, then the hypotheses of the Corollary are automatically satisfied.

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In other words a variety V is classically ideal determined is there is an $n \in \mathbb{N}$, binary terms $d_1, \dots, d_n$ and a $n + 1$-ary term t such that

$$d_i(x, x) \approx 0 \quad i = 1. \dots, n$$
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Proposition

A classically ideal determined variety is 0*-regular and congruence permutable, hence ideal determined.*

Varieties that are 0-regular and congruence permutable have been described in [1]:

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Theorem

[1] For a variety V *the following are equivalent:*

1. V *is* 0*-regular and congruence permutable;*
2. *there is an* $n \in \mathbb{N}$*, an* $n+2$*-ary term* p *and binary terms* $d_1, \ldots, d_n$ *such that*

$$\begin{aligned} &d_i(x,x) \approx 0 \qquad i = 1, \ldots, n \\ &p(x,y,0,\ldots,0) \approx y \\ &p(x,y,d_1(x,y),\ldots,d_n(x,y)) \approx x \end{aligned}$$

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A variety V is **classically** 0-**regular** if for all $\mathbf{A} \in$ V and $\mathbf{S}, \mathbf{T} \in \mathrm{Cs}(\mathbf{A})$ if $0/\mathbf{S} = 0/\mathbf{T}$ and $S^{\Delta} \subseteq T$ and $T^{\Delta} \subseteq S$, then $\mathbf{S} = \mathbf{T}$. Clearly every classically 0-regular variety is 0-regular as well.

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A variety V is 0-**coherent** if for all $\mathbf{A} \in \mathrm{B}$, for all $\theta \in \mathrm{Con}(\mathbf{A})$ and for all $\mathbf{B} \leq \mathbf{A}$, if $0/\theta \subseteq B$, then B is a union of θ-blocks.

Theorem

[9] For a variety V *the following are equivalent:*

1. V *is classically ideal determined;*
2. V *is classically* 0*-regular;*
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3. V *is* 0*-coherent.*

The proof is rather technical, albeit similar to the ones we have already seen. Therefore we omit it.

An example

Let $\mathbf{A} = \{0, a, b, c\}$; on A we define the following operations:

- $d(x, y)$ is a binary operation whose table is

d	0	a	b	c
0	0	a	b	c
a	a	0	c	a
b	b	c	0	a
c	c	a	a	a

- $t(x, y, z)$ is a ternary operation defined by

$$g(x, y, z) = \begin{cases} x, & \text{if } z = 0; \\ y, & \text{if } d(x, y) = z; \\ z, & \text{otherwise.} \end{cases}$$

We spare the tedious verification that in **A** the following equations hold:

$$
\begin{aligned}
d(x, x) &\approx 0 \\
t(x, y, 0) &\approx x \\
t(x, y, d(x, y)) &\approx x.
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However it is easy to check that the partition $\{\{a,b\},\{0,c\}\}$ induces a congruence on **A** and that $\{0,a,c\}$ is the universe of a subalgebra of **A**.

So V(**A**) is not 0-coherent and thus it is not classically ideal determined.

Strongly subtractive varieties

Let V be a subtractive variety, witness $s(x, y)$; we say that V is **strongly subtractive** if for all $\mathbf{A} \in V$ and $I \in \mathrm{Id}(\mathbf{A})$ the relation

$$(a, b) \in I^* \qquad \text{if and only if} \qquad s(b, a) \in I$$

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Note that I^* is necessarily reflexive and this allows us to prove a result similar to the one by Werner [10] in congruence permutable variety.

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Proposition

Let $\mathbf{A}$ *be an algebra in a subtractive variety and* $I \in \mathrm{Id}(\mathbf{A})$*; then the following are equivalent:*

1. I^* *is a congruence;*
2. I^* *is a subalgebra of* $\mathbf{A} \times \mathbf{A}$.

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If $\mathbb{V}$ *is* 0*-regular and strongly subtractive then it is classically ideal determined.*

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Let $\mathbf{F} = \mathbf{F}_{\mathsf{V}}(x, y)$ and let $\theta = \vartheta_{\mathbf{F}}(x, y)$ and let $I = 0/\theta$; since V is strongly subtractive the relation

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Now $u \in 0/I^*$ if and only if $u \in I = 0/\theta$; as V is 0-regular, $\theta = I^*$ and in particular $(x, y) \in I^*$.

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Therefore (x, y) belongs to the subalgebra of $\mathbf{F}^2$ generated by $\{(y, y)\} \cup \{(s(u, v), 0) : s(u, v) \in 0/\theta\}$; since the lattice of subalgebras is algebraic, there is an n and $u_1, \ldots, u_n, v_1, \ldots, v_n \in F$ such that (x, y) belongs to the subalgebra generated by

$$\{(y, y)\} \cup \{(s(u_i(x, y), v_i(x, y)), 0) : i = 1, \ldots, n\}.$$

Let now $d_i := s(u_i, v_i)$ for $i = 1, \ldots, n$; then there is an $n+1$-ary term t such that

$$(x, y) = t((y, y), (d_1(x, y), 0), \ldots, (d_n(x, y), 0))$$

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Next let φ be endomorphism of $\mathbf{F}$ sending $x, y \longmapsto x$; then $\theta \subseteq \ker(\varphi)$. So

$$d_i(x, x) = s(u_i(x, x), v_i(x, x)) \; \theta \; s(u_i(x, y), v_i(x, y)) \; \theta \; 0$$

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Hence V is classically ideal determined.

Many examples of strongly subtractive varieties come either from classical algebras or the algebraization of logical systems. In the first class we quote groups, rings, **R**-modules and more generally associative algebras over a ring.

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We conclude with a characterization of strongly subtractive varieties.

Theorem

For a variety V *the following are equivalent:*

1. V *is strongly subtractive witness* $s(x, y)$*;*
2. V *is subtractive and for all n-ary basic operation* f *of* V *there is an 3n-ary term* r_f *such that*

$$s(f(\mathrm{x}), f(\mathrm{y})) \approx r_f(\mathrm{x}, \mathrm{y}, s(x_1, y_1), \ldots, s(x_n, y_n))$$

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Assume (1) and let f be an n-ary operation. Consider the free algebra $\mathbf{F}$ in V generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ and let I be the ideal generated by $\{s(x_i, y_i), i = 1, \ldots, n\}$.

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Since V is subtractive, the is a congruence θ of **F** with $I = 0/\theta$. Since V is strongly subtractive, $(f(\vec{x}), f(\vec{y})) \in \theta^*$. Hence $(s(f(\vec{x}), f(\vec{y})), 0) \in \theta$ and so $s(f(\vec{x}), f(\vec{y})) \in I$. From here a standard argument yields a term r_f with the desired properties. Thus we can conclude that V satisfies (2).

Conversely assume (2) and let $\mathbf{A} \in \mathsf{V}$ and $\theta \in \mathrm{Con}(\mathbf{A})$.

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Let φ be the subalgebra generated by θ^*; then if $(a, 0) \in \varphi$ there are $a_1, \ldots, a_n, b_1, \ldots, b_n$ with $(a_i, b_i) \in \theta^*$ and a term u such that $u(a_1, \ldots, a_n) = a$ and $u(b_1, \ldots, b_n) = 0$.

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since $s(a_i, b_i) \in 0/\theta$ for all i we may conclude that $a \in 0/\theta$ and hence that $(a, 0) \in \theta^*$.

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since $s(a_i, b_i) \in 0/\theta$ for all i we may conclude that $a \in 0/\theta$ and hence that $(a, 0) \in \theta^*$.

This implies that $0/\varphi = 0/\theta^*$ for all i. The fact that this implies that θ^* is a subalgebra is left as an exercise to the reader.

THANK YOU!

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