Ideals in universal algebra I: Definitions and first results

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What is an ideal?

Given an algebra $\mathbf{A}$ an ideal is an "interesting subset" of the universe A, that may or may not be a subalgebra of $\mathbf{A}$; an example of the first kind is a normal subgroup of the group and of the second kind is an ideal of a commutative ring (we follow the modern *dictum* that every ring has a multiplicative unit)

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- an ideal must have a simple algebraic definition;
- ideals must be closed under arbitrary intersections, so that a closure operator can be defined in which the ideals are exactly the closed sets; this gives raise to an algebraic lattice whose elements are exactly the ideals;
- ideals must convey meaningful information on the structure of the algebra.

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An n ideal on a set X is an ideal (in the lattice sense) on the Boolean algebra of subsets of X. There also a significant difference between ideals on lattices and ideals on Boolean algebras; in Boolean algebras an ideal is always the 0-class of a suitable congruence of the algebra (really, of exactly one congruence), while this is not true in general for lattices.

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As a matter of fact, identifying the class of (lower bounded) lattices in which every ideal is the 0-class of a congruence is a difficult problem which is still unsolved, up to our knowledge. Of course the same property is shared by normal subgroups of group and (two-sided) ideals of a ring (since they are both congruence kernels).

The problem of connecting ideals of general algebras to congruence classes has been foreshadowed in [5] but really tackled by A. Ursini in his seminal paper [6]. Later, from the late 1980's to the late 1990's, A. Ursini and the author published a long series of papers on the subject ([7], [2], [3], [4], [1]); the theory developed in those papers will constitute the basis of these notes.

We postulated that an ideal must have a simple algebraic definition; as imprecise as this concept might be, in our context there is a natural path to follow. Given a type (a.k.a. a signature) σ we can consider the σ-**terms** (i.e. the elements of $\mathbf{T}_\sigma(\omega)$, the absolutely free countably generated algebra of type σ); a term is denoted by $p(x_1, \dots, x_n)$ to emphasize the variable involved and we will use the vector notation $\vec{x}$ for $x_1, \dots, x_n$.

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Let Γ be a set of σ-terms; we will divide the (finite) set of variables $z_1, \dots, z_{n+m}$ of each term in two subsets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ so that every term in Γ can be expressed as $p(\vec{x}, \vec{y})$ and we allow $n = 0$, while m must always be at least 1.

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Moreover we ask that Γ be closed under *composition on* $\vec{y}$; this means that if $p(\vec{x}, \vec{y} \in \Gamma)$, $\vec{y} = (y_1, \dots, y_m)$ and $p_1(\vec{x}^1, \vec{y}^1), \dots, p_m(\vec{x}^m, \vec{y}^m) \in \Gamma$, then

$$p(\vec{x}, p_1(\vec{x}^1, \vec{y}^1), \dots, p_m(\vec{x}^m, \vec{y}^m)) \in \Gamma.$$

If **A** has type σ a Γ-ideal of **A** is an $I \subseteq A$ such that for any $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_m \in I$ and $p(x, y) \in \Gamma$, $p(\vec{a}, \vec{b}) \in I$.

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Lemma

Let σ be any type, Γ a set of σ-terms closed under composition on $\vec{y}$ *and $\mathbf{A}$ an algebra of type σ. Then*

1. *the Γ-ideals of $\mathbf{A}$ are closed under arbitrary intersections;*
2. *the Γ-ideal generated by $X \subseteq A$, i.e. the intersection of all the Γ-ideals containing X, is*

$$(X)^\Gamma_{\mathbf{A}} = \{p(\vec{a}, \vec{b}) : \vec{a} \in A, \vec{b} \in X, p(\vec{x}, \vec{y}) \in \Gamma\};$$

3. *the Γ-ideals of $\mathbf{A}$ form an algebraic lattice $\mathrm{Id}^\Gamma(\mathbf{A})$.*

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At this level of generality we cannot say much more; if the type however contains a constant we can get a more focused definition.

Let V be a variety whose type contains a constant which will denote by 0; a $\mathsf{V}, 0$-**ideal term in** $y_1, \dots, y_m$ is a term $p(\vec{x}, \vec{y})$ such that

$$\mathsf{V} \vDash p(\vec{x}, 0, \dots, 0) \approx 0.$$

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Let $ID_{\mathsf{V},0}$ be the set of all $\mathsf{V}, 0$-ideal terms in V; a $\mathsf{V}, 0$-**ideal** I of $\mathbf{A} \in \mathsf{V}$ is a $ID_{\mathsf{V},0}$-ideal of $\mathbf{A}$.

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The set $\mathrm{Id}_{\mathsf{V},0}(\mathbf{A})$ of $\mathsf{V}, 0$-ideals of $\mathbf{A}$ is an algebraic lattice.

It is also evident that for any $\theta \in \mathrm{Con}(\mathbf{A})$, $0/\theta$ is a $\mathsf{V}, 0$-ideal of $\mathbf{A}$: if $p(\vec{x}, \vec{y}) \in ID_{\mathsf{V},0}$, $\vec{a} \in A$ and $\vec{b} \in 0/\theta$ then

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Similarly $(X)_{\mathbf{A}}$ for $X \subseteq A$ will denote the $\boldsymbol{V}(\mathbf{A}), 0$-ideal of $\mathbf{A}$ generated by X.

Proposition

For any algebra $\mathbf{A}$ *with* 0, $\mathrm{I}(\mathbf{A})$ *is isomorphic with the ideal lattice of* $\mathrm{CI}(\mathbf{A})$ *(semilattice ideals in the usual sense).*

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Proposition

Let $\mathbf{R}$ *be a subalgebra of* $\mathbf{A} \times \mathbf{A}$ *such that* $\pi_2(\mathbf{R}) = A$, *where* π_2 *denotes the second projection. If* $K \in \mathrm{I}(\mathbf{R})$ *and* $I \in \mathrm{I}(\mathbf{A})$, *then*

$$(I)_K = \{b \in B : \text{for some } a \in I, (a, b) \in K\}$$

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In particular $\mathbf{R}$ may be a subdirect product or a reflexive subalgebra of $\mathbf{A} \times \mathbf{A}$.

Proposition

Let $\theta \in \mathrm{Con}(\mathbf{A})$. *There is a one-to-one correspondence (which is in fact a complete lattice isomorphism) between the ideals* I *of* $\mathbf{A}$ *such that* $0/\theta \subseteq I$ *and the ideals of* $\mathbf{A}/\theta$. *The correspondence is*

$$I \longmapsto I/\theta = \{a/\theta : a \in I\}$$

Normal ideals

We say that V has **normal** $\mathsf{V}, 0$**-ideals** if for all $\mathbf{A} \in \mathsf{V}$ for all $I \in \mathrm{Id}_{\mathsf{V},0}(\mathbf{A})$ there is a $\theta \in \mathrm{Con}(\mathbf{A})$ with $I = 0/\theta$.

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If V has normal V, 0-ideals then of course

$$\mathrm{Id}_0(\mathbf{A}) = \mathrm{Id}_{\mathsf{V},0}(\mathbf{A}) = \{0/\theta : \theta \in \mathrm{Con}(\mathbf{A})\}$$

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so we can simply talk about 0-ideals of $\mathbf{A}$ without specifying the variety.

Observe that the variety of pointed (by 0) sets has normal 0-ideals, so we can hardly expect any nice structural theorem for varieties with 0-normal ideals.

Theorem

[1] For any algebra $\mathbf{A}$ *the following are equivalent:*

1. $\mathbf{A}$ *has normal ideals;*
2. $(X)_{\mathbf{A}} = 0/\vartheta_{\mathbf{A}}(X)$ *for any* $X \subseteq A$*;*
3. $I/\vartheta_{\mathbf{A}}(J) = I \vee J$ *for any* $I, J \in \mathrm{I}(\mathbf{A})$*;*
4. $I/\vartheta_{\mathbf{A}}(J) = J/\vartheta_{\mathbf{A}}(I)$ *for any* $I, J \in \mathrm{I}(\mathbf{A})$*;*
5. *the mapping from* $\mathrm{I}(\mathbf{A})$ *to* $\mathrm{Con}(\mathbf{A})$ *sending* $I \longmapsto \vartheta_{\mathbf{A}}(I)$ *is one-to-one;*
6. *the mapping from* $\mathrm{Con}(\mathbf{A})$ *to* $\mathrm{I}(\mathbf{A})$ *sending* $\theta \longmapsto 0/\theta$ *is onto.*

Ideal irreducibility

In analogy to subdirect irreducibility let us define an algebra **A** to be **ideal irreducible** if for any family $(I_\lambda)_{\lambda\in\Lambda}$ of ideals of **A**, if $\bigcap_{\lambda\in\Lambda} I_\lambda = (0)$, then, for some λ, $I_\lambda = (0)$; the concept of **finitely ideal irreducible** is defined in an obvious way.

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An algebra **A** is ideal irreducible if and only if there is a minimal nonzero ideal, which then must be principal, generated by a **monolithic element** a (namely $a \neq 0$ and $a \in I$ for any nonzero $I \in \mathrm{I}(\mathbf{A})$).

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Proposition

Assume $\mathrm{N}(\mathbf{A}) = \mathrm{I}(\mathbf{A})$. *If* $a \neq 0$, $a \in A$, *then there is a* $\theta \in \mathrm{Con}(\mathbf{A})$ *such that* a/θ *is monolithic in* $\mathbf{A}/\theta$.

Proof.

In fact, let H be maximal in $\{I \in \mathrm{I}(\mathbf{A}) : a \notin I\}$, via Zorn Lemma. Let $H = 0/\theta$; if $J \in \mathrm{I}(\mathbf{A}/\theta)$ then $J = I/\theta$ with $H \subseteq I$. Take $b/\theta \in J$ with $b/\theta \neq 0/\theta$; then $b \in I - H$. Hence, by the maximality of H, $a \in I$, Hence $a/\theta \in J$. □

Proposition

Let $\theta \in \mathrm{Con}(\mathbf{A})$*. If* $a \notin 0/\theta$ *and* $\mathrm{N}(\mathbf{A}/\theta) = \mathrm{I}(\mathbf{A}/\theta)$*, then there is a* $\varphi \in \mathrm{Con}(\mathbf{A})$*,* $\varphi \supseteq \theta$ *such that* a/φ *is monolithic in* $\mathbf{A}/\varphi$*.*

Proof.

In fact, apply the previous proposition to $\mathbf{A}/\theta$ and recall that congruences of $\mathbf{A}/\theta$ corresponds to congruences of $\mathbf{A}$ containing θ. We then get a congruence $\varphi \supseteq \theta$ such that $(a/\theta)/(\varphi/\theta)$ is monolithic in $(\mathbf{A}/\theta)/(\varphi/\theta)$, namely a/φ is monolithic in $\mathbf{A}/\varphi$. □

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An algebra $\mathbf{A} \in$ V is said to be 0-**permutable**, or to have 0-**permutable congruences** if for all $a \in A$ and $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$, if $(a, 0) \in \theta \circ \varphi$, then $(a, 0) \in \varphi \circ \theta$.

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An algebra $\mathbf{A}$ is called **ideal-coherent** if, for any $I \in \mathrm{I}(\mathbf{A})$ and $\theta \in \mathrm{Con}(\mathbf{A})$, $0/\theta \subseteq I$ yields that I is a union of θ-blocks.

Theorem

[1] For a variety V *the following are equivalent:*

1. *for all* $\mathbf{A} \in \mathsf{V}$ *and* $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$ *we have* $0/(\theta \vee \varphi) = 0/(\theta \circ \varphi)$;
2. *every algebra in* V *has* 0*-permutable congruences;*
3. V *is subtractive ;*
4. *there is a ternary term* $w(x, y, z)$ *of* V *such that*
$$w(x, y, y) \approx x \qquad w(x, x, 0) \approx 0$$
hold in V*;*
5. *there exists a positive integer* m*, binary terms* $d_1(x, y), \ldots, d_m(x, y)$ *and an* $m + 3$*-ary term* $q(x_1, \ldots, x_{m+3})$ *of* V *such that*
$$d_i(x, x) \approx 0 \quad \text{for} \quad i = 1, \ldots, m$$
$$q(x, y, 0, 0, \ldots, 0) \approx 0$$
$$q(x, y, y, d_1(x, y), \ldots, d_m(x, y)) \approx x$$
hold in V*;*
6. V *is* **ideal-coherent***;*
7. *for all* $\mathbf{A} \in \mathsf{V}$*, the mapping* $\mathrm{Con}(\mathbf{A}) \longrightarrow \mathrm{I}(\mathbf{A})$ *defined by* $\theta \longmapsto 0/\theta$ *is a complete and onto lattice homomorphism.*

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Then $(x, 0) \in \varphi \circ \theta$ and so the usual Mal'cev argument yields a term $s(x, y)$ satisfying the equations.

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If (3) holds we set $w(x, y, z) := s(x, s(y, z))$ and we check that the equations in (4) hold.

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If (3) holds we set $w(x, y, z) := s(x, s(y, z))$ and we check that the equations in (4) hold.

Finally if (4) holds and $(a, 0) \in \theta \circ \varphi$ then there is a b such that $a\ \theta\ b\ \varphi\ 0$; hence

$$a = w(a, 0, 0)\ \varphi\ w(a, b, 0)\ \theta\ w(b, b, 0) = 0.$$

So $(a, 0) \in \theta \circ \varphi$. This implies that $0/\theta \circ \varphi = 0/\varphi \circ \theta$ and also very easily (1).

(3) implies (5) which is equivalent to (6)

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Let $I \in \mathrm{I}(\mathbf{A})$, $\theta \in \mathrm{Con}(\mathbf{A})$ and $0/\theta \subseteq I$. Let $v \in I$ with $(u, v) \in \theta$; then for all $i \leq m$ we have $d_i(u, v)\ \theta\ d_i(v, v) = 0$, hence $d_i(u, v) \in I$. Note that $q(x, y, \vec{z})$ is an ideal term in $\vec{z}$, so we must have

$$u = q(u, v, v, d_1(u, v), \ldots, d_m(u, v)) \in I.$$

Assume that V is ideal-coherent and look at $\mathbf{F}_\mathsf{V}(x, y)$. Let θ_f be the congruence associated with the endomorphism of $\mathbf{F}_\mathsf{V}(x, y)$ defined by $f(x) = f(y) = x$ and $f(0) = 0$. Let

$$I = \mathrm{Id}_\mathbf{A}(\{y\} \cup \{d(x, y) \in \mathbf{F}_\mathsf{V} x, y : d(x, y) \in 0/\theta_f\}).$$

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$$u = q(u, v, v, d_1(u, v), \dots, d_m(u, v)) \in I.$$

Assume that V is ideal-coherent and look at $\mathbf{F}_{\mathsf{V}}(x, y)$. Let θ_f be the congruence associated with the endomorphism of $\mathbf{F}_{\mathsf{V}}(x, y)$ defined by $f(x) = f(y) = x$ and $f(0) = 0$. Let

$$I = \mathrm{Id}_{\mathbf{A}}(\{y\} \cup \{d(x, y) \in \mathbf{F}_{\mathsf{V}} x, y : d(x, y) \in 0/\theta_f\}).$$

Then clearly $0/\theta_f \subseteq I$, $y \in I$ and $(x, y) \in \theta_f$. Thus ideal-coherency yields $x \in I$.

Then there is an ideal term $t(\vec{x}, y, \vec{z})$ in $y \cup \vec{z}$ and $d_1(x, y), \ldots, d_m(x, y) \in 0/\theta_f$ with

$$t(\vec{u}, y, d_1(x, y), \ldots, d_m(x, y)) = x.$$

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Since any $u_j = u_j(x, y)$ we do get $m + 3$-ary term by setting

$$q(x, y, y, z_1, \dots, z_m) = t(\vec{u}, y, z_1, \dots, z_m).$$

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But then $q(x, y, 0, 0, \ldots, 0) = 0$, since q is an ideal term in $y \cup \vec{z}$. As showed above $q(x, y, y, d_1(x, y), \ldots, d_m(x, y)) = x$ and finally, for all i, $d_i(x, y) \in 0/\theta_f$ that yields $d_i(x, x) = f(d_i(x, y)) = f(0) = 0$. Therefore (5) and (6) are equivalent.

(5) implies (2)

Assume again (5). Let $\mathbf{A} \in \mathsf{V}$, $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$ and $a \in 0/(\theta \circ \varphi)$.

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Then there is a $b \in A$ with $(0, b) \in \theta$ and $(b, a) \in \varphi$.

Hence we get $d_i(a, b)\ \varphi\ d_i(b, b)\ \varphi\ 0$ for all i. So

$$\begin{aligned} 0 &= q(a, b, 0, 0, \dots, 0) \\ &\varphi\ q(a, b, 0, d_1(a, b), \dots, d_m(a, b)) \\ &\theta\ q(a, b, b, d_1(a, b), \dots, d_m(a, b)) = a, \end{aligned}$$

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$$\begin{aligned} 0 &= q(a, b, 0, 0, \dots, 0) \\ &\varphi \; q(a, b, 0, d_1(a, b), \dots, d_m(a, b)) \\ &\theta \; q(a, b, b, d_1(a, b), \dots, d_m(a, b)) = a, \end{aligned}$$

hence $(0, a) \in \varphi \circ \theta$ and V is 0-permutable. Therefore (5) implies (2).

(3) implies (7)

Assume now (3). Let $\mathbf{A} \in \mathsf{V}$, $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$ and $a \in 0/(\theta \vee \varphi)$. Then there are $a_1, \dots, a_n \in A$ with

$$a \ \theta \ a_1 \ \varphi \ \alpha_2 \ \theta \ \dots \ \theta \ a_n \ \varphi \ 0.$$

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$$a \ \theta \ a_1 \ \varphi \ \alpha_2 \ \theta \ \dots \ \theta \ a_n \ \varphi \ 0.$$

Let us set $t(x, y, z,) = s(x, s(s(x, y), z,))$ and let us induct on n. If $n = 1$ then $a \ \theta \ a_1 \ \varphi \ 0$. Hence $s(a, a_1) \ \theta \ 0$, therefore

$$a = t(a, a_1, s(a, a_1)) \in 0/(\theta \vee \varphi)$$

being $t(x, y, z)$ an ideal term in y, z.

Let now assume the statement true for n and let

$$a \ \theta \ a_1 \ \varphi \ \alpha_2 \ \theta \ \ldots \ \theta \ a_n \ \varphi \ a_{n+1} \ \theta \ 0.$$

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Then $s(a, a_{n+1}) \ \varphi \ s(a, a_n) \ \theta \ \ldots \ \theta \ s(a, a) = 0$, so, by induction hypothesis, $s(a, a_{n+1}) \in 0/\theta \vee 0/\varphi$. But since $a_{n+1} \in 0/\theta$ we get again

$$a = t(a, a_{n+1}, s(a, a_{n+1})) \in 0/\theta \vee 0/\varphi.$$

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The case $a_{n+1} \ \varphi \ 0$ is totally similar hence we conclude that $0/(\theta \vee \varphi) \subseteq 0/\theta \vee 0/\varphi$.

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For the converse let $a \in 0/\theta \vee 0/\varphi$; then there are an ideal term $p(\vec{x}, \vec{y}, \vec{z})$ in $\vec{y} \cup \vec{z}$, $\vec{a} \in A$, $\vec{u} \in 0/\theta$ and $\vec{v} \in 0/\varphi$ with $a = p(\vec{a}, \vec{u}, \vec{v})$.

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$$a \ \theta \ a_1 \ \varphi \ \alpha_2 \ \theta \ \dots \ \theta \ a_n \ \varphi \ a_{n+1} \ \theta \ 0.$$

Then $s(a, a_{n+1}) \ \varphi \ s(a, a_n) \ \theta \ \dots \ \theta \ s(a, a) = 0$, so, by induction hypothesis, $s(a, a_{n+1}) \in 0/\theta \vee 0/\varphi$. But since $a_{n+1} \in 0/\theta$ we get again

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If we now set $b = p(\vec{a}, \vec{u}, 0, \dots, 0)$ we get $(b, 0) \in \theta$ and $(a, b) \in \varphi$; therefore $a \in 0/(\theta \vee \varphi)$.

Let now assume the statement true for n and let

$$a \ \theta \ a_1 \ \varphi \ \alpha_2 \ \theta \ \ldots \ \theta \ a_n \ \varphi \ a_{n+1} \ \theta \ 0.$$

Then $s(a, a_{n+1}) \ \varphi \ s(a, a_n) \ \theta \ \ldots \ \theta \ s(a, a) = 0$, so, by induction hypothesis, $s(a, a_{n+1}) \in 0/\theta \vee 0/\varphi$. But since $a_{n+1} \in 0/\theta$ we get again

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The case $a_{n+1} \ \varphi \ 0$ is totally similar hence we conclude that $0/(\theta \vee \varphi) \subseteq 0/\theta \vee 0/\varphi$.

For the converse let $a \in 0/\theta \vee 0/\varphi$; then there are an ideal term $p(\vec{x}, \vec{y}, \vec{z})$ in $\vec{y} \cup \vec{z}$, $\vec{a} \in A$, $\vec{u} \in 0/\theta$ and $\vec{v} \in 0/\varphi$ with $a = p(\vec{a}, \vec{u}, \vec{v})$.

If we now set $b = p(\vec{a}, \vec{u}, 0, \ldots, 0)$ we get $(b, 0) \in \theta$ and $(a, b) \in \varphi$; therefore $a \in 0/(\theta \vee \varphi)$.

Hence we conclude that (3) implies (7).

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That (7) implies (2) follows from the fact that $a \in 0/(\theta \vee \varphi)$ implies $a \in 0/\theta \vee 0/\varphi$, hence $(a, 0) \in \theta \circ \varphi$.

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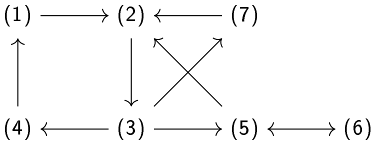

the term $u(x, y, z)$

If $s(x, y)$ witnesses subtractivity for V we define a ternary term $u(x, y, z) := s(x, s(s(x, y), z))$ and we observe note that the following identities hold:

1. $u(x, y, s(x, y)) \approx x$
2. $u(x, 0, 0) \approx 0$
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This term is very useful; the first application is yet other equivalent conditions for subtractivity that one may add to the ones in the previous theorem.

Proposition

For a variety V *with* 0 *the following are equivalent:*

1. V *is subtractive;*
2. *there is a binary term* $t(x,y)$ *of* V *such that* $t(x,x) \approx 0$ *and for any* $\mathbf{A} \in \mathsf{V}$ *and* $a, b \in A$
$$a \in (b)_{\mathbf{A}} \vee (t(a,b))_{\mathbf{A}};$$
3. *there is a binary term* $t(x,y)$ *of* V *such that* $t(x,x) \approx 0$ *and for any* $\mathbf{A} \in \mathsf{V}$ *and* $a \in A$
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$$a \in (t(a,0))_{\mathbf{A}};$$

The proof is straightforward an it is left as an exercise.

Another consequence is that subtractive varieties have normal ideals and the quickest way to show it is to use the so called *Mal'cev criterion* whose proof is left again as an exercise,

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Lemma

(Mal'cev) Let $\mathbf{A}$ *be an algebra and let* $I \subseteq A$*,* I *non empty; then the following are equivalent*

1. *there is a* $\theta \in \mathrm{Con}(\mathbf{A})$ *with* $I = a/\theta$ *for some* $a \in A$*;*
2. *for every unary polynomial* $g(x)$ *of* $\mathbf{A}$ *if* $a, b, g(a) \in I$ *then* $g(b) \in I$*.*

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Every subtractive variety V *has normal ideals.*

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Proof.

Let $\mathbf{A} \in \mathsf{V}$ and $I \in \mathrm{I}(\mathbf{A})$. Let $g(x)$ be a unary polynomial of $\mathbf{A}$; then there is an $n+1$-term $t(\vec{y}, x)$ and $\vec{a} \in A$ with $t(\vec{a}, x) = g(x)$.

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Let $a, b, g(a) \in I$ and observe that $s(t(\vec{y}, x_1), t(\vec{y}, x_2))$ is an ideal term in x_1, x_2.

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Let $a, b, g(a) \in I$ and observe that $s(t(\vec{y}, x_1), t(\vec{y}, x_2))$ is an ideal term in x_1, x_2.

Therefore $s(g(b), g(a)) \in I$; as $g(a) \in I$ we get that $g(b) = u(g(b), g(a), s(g(b), g(a))) \in I$.

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Let $a, b, g(a) \in I$ and observe that $s(t(\vec{y}, x_1), t(\vec{y}, x_2))$ is an ideal term in x_1, x_2.

Therefore $s(g(b), g(a)) \in I$; as $g(a) \in I$ we get that
$g(b) = u(g(b), g(a), s(g(b), g(a))) \in I$.

By Mal'cev criterion I is a congruence class, hence it is $0/\theta$ for some $\theta \in \mathrm{Con}(\mathbf{A})$. □

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Lemma

If V *is subtractive,* $\mathbf{A} \in \mathsf{V}$ *and* $I, J \in \mathrm{I}(\mathbf{A})$ *then*

$$I \vee J = \{u(a, b, c) : a \in A, b \in I, c \in J\}.$$

Proof.

Let $K = \{u(a, b, c) : a \in A, b \in I, J \in J\}$; a generic element $b \in (K)_{\mathbf{A}}$ is of the form

$$p(\vec{a}, u(d_1, e_1, f_1), \ldots, u(d_m, e_m, f_m))$$

where $p(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$, $\vec{a}, d_1, \ldots, d_m \in A$, $e_1, \ldots, e_m \in I$, $f_1, \ldots, f_m \in J$.

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Let

$$c = p(\vec{a}, u(d_1, e_1, 0), \dots, u(d_m, e_m, 0));$$

hence $c \in I$.

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Let

$$c = p(\vec{a}, u(d_1, e_1, 0), \dots, u(d_m, e_m, 0));$$

hence $c \in I$.

Since

$$s(p(\vec{x}, u(z_1, y_1, w_1), \dots, u(z_k, y_k, w_k))), p(\vec{x}, u(z_1, y_1, 0), \dots, u(z_k, y_k, 0))$$

is an ideal term in $\vec{w}$ we have that $s(b, c) \in J$. It follows that $b = u(b, c, s(b, c)) \in K$. Hence $(K)_{\mathbf{A}} \subseteq K$ and thus equality holds.

Proof.

Let $K = \{u(a,b,c) : a \in A, b \in I, J \in J\}$; a generic element $b \in (K)_{\mathbf{A}}$ is of the form

$$p(\vec{a}, u(d_1, e_1, f_1), \ldots, u(d_m, e_m, f_m))$$

where $p(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$, $\vec{a}, d_1, \ldots, d_m \in A$, $e_1, \ldots, e_m \in I$, $f_1, \ldots, f_m \in J$.

Let

$$c = p(\vec{a}, u(d_1, e_1, 0), \ldots, u(d_m, e_m, 0));$$

hence $c \in I$.

Since

$$s(p(\vec{x}, u(z_1, y_1, w_1), \ldots, u(z_k, y_k, w_k))), p(\vec{x}, u(z_1, y_1, 0), \ldots, u(z_k, y_k, 0))$$

is an ideal term in $\vec{w}$ we have that $s(b,c) \in J$. It follows that $b = u(b, c, s(b,c)) \in K$. Hence $(K)_{\mathbf{A}} \subseteq K$ and thus equality holds.

So K is an ideal containing I, J and it is clearly the smallest. This proves the thesis. □

Finally:

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If V *is subtractive, then for all* $\mathbf{A} \in \mathsf{V}$, $\mathrm{I}(\mathbf{A})$ *is a modular lattice.*

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Let $I, J, H \in \mathrm{I}(\mathbf{A})$ and suppose that $I \subseteq J$, $I \vee H = J \vee H$ and $I \cap H = J \cap H$.

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Proof.

Let $I, J, H \in \mathrm{I}(\mathbf{A})$ and suppose that $I \subseteq J$, $I \vee H = J \vee H$ and $I \cap H = J \cap H$.

If $a \in J$, then $a \in I \vee H$, so by (the proof of) Lemma 12 for some $c \in I$ we have that $s(a, c) \in H$. As $I \subseteq J$, $c \in J$, and thus $s(a, c) \in J \cap H = I \cap H$.

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If V *is subtractive, then for all* $\mathbf{A} \in \mathsf{V}$, $\mathrm{I}(\mathbf{A})$ *is a modular lattice.*

Proof.

Let $I, J, H \in \mathrm{I}(\mathbf{A})$ and suppose that $I \subseteq J$, $I \vee H = J \vee H$ and $I \cap H = J \cap H$.

If $a \in J$, then $a \in I \vee H$, so by (the proof of) Lemma 12 for some $c \in I$ we have that $s(a, c) \in H$. As $I \subseteq J$, $c \in J$, and thus $s(a, c) \in J \cap H = I \cap H$.

In particular $s(a, c) \in I$ and thus $a = u(a, c, s(a, c)) \in I$. So $J \subseteq I$ and hence $I = J$; this proves modularity of $\mathrm{I}(\mathbf{A})$. □

THANK YOU!

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