Ideals in universal algebra

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Abstract

Given an algebra $\mathbf{A}$ an ideal is an "interesting subset" of the universe A, that may or may not be a subalgebra of $\mathbf{A}$; an example of the first kind is a normal subgroup of the group and of the second kind is an ideal of a commutative ring[1]. Now defining what "interesting" means is largely a matter of taste; however there is a large consensus among the practitioners of the field that:

- an ideal must have a simple algebraic definition;
- ideals must be closed under arbitrary intersections, so that a closure operator can be defined in which the ideals are exactly the closed sets; this gives raise to an algebraic lattice whose elements are exactly the ideals;
- ideals must convey meaningful information on the structure of the algebra.

The three points above are all satisfied by classical ideals on lattices and of course by ideals on a set X, where we interpret a set as an algebra in which the set of fundamental operations is empty. We have however to be careful here; an ideal on a set X is an ideal (in the lattice sense) on the Boolean algebra of subsets of X. There also a significant difference between ideals on lattices and ideals on Boolean algebras; in Boolean algebras an ideal is always the 0-class of a suitable congruence of the algebra (really, of exactly one congruence), while this is not true in general for lattices. As a matter of fact, identifying the class of (lower bounded) lattices in which every ideal is the 0-class of a congruence is a difficult problem which is still unsolved, up to our knowledge.

The problem of connecting ideals of general algebras to congruence classes has been foreshadowed in [5] but really tackled by A. Ursini in his seminal paper [6]. Later, from the late 1980's to the late 1990's, A. Ursini and the author published a long series of papers on the subject ([7], [2], [3], [4], [1]); the theory developed in those papers will constitute the basis of the course.

In details first we will get acquainted with the general theory of ideals, then we will explore some particular and interesting cases in which the theory can be strengthened. Finally, if time allows, we will see some applications if the theory to specific topics in the general universal algebraic settings.

[1] we follow the modern *dictum* that every ring has a multiplicative unit...

References

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