On fuzzy Galois connections between L-fuzzy posets

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Abstract

We consider properties of fuzzy Galois connections between L-fuzzy posets for complete residuated lattices L and show their properties by operator-based algebraic methods. We also give a simple condition for the existence of right adjoint map.

For given two partially ordered sets (simply called posets) $(X, \leq_X)$ and $(Y, \leq_Y)$, these sets have some relationship between them when there are maps such as order-preserving maps, injective maps. Especially, a pair (f, g) of maps, called a pair of Galois connection, is very important notion for posets. Roughly speaking, this indicates that if there is a Galois connection between posets, then a certain property in one poset is "similarly" reflected in another poset and vice versa. Therefore, Galois connections arise very frequently in many fields on mathematics, computer science and so on, and their results make us the situation much more easily understood.

Let $(X, \leq_X)$, $(Y, \leq_Y)$ be posets and $f : X \to Y,\ g : Y \to X$ maps. A pair (f, g) of maps is called a Galois connection if $f(x) \leq_Y y$ if and only if $x \leq_X g(y)$ for all $x \in X, y \in Y$. It is easy to show that (f, g) is a Galois connection if and only if both f, g are order-preserving and $x \leq_X gf(x)$ and $fg(y) \leq_Y y$ for all $x \in X, y \in Y$. In this case, two maps f, g are called the left and the right adjoint, respectively.

We have a familiar example of Galois connection in the classical set theory. Let X, Y be non-empty sets and $f : X \to Y$ a map. It is obvious that both $(2^X, \subseteq)$ and $(2^Y, \subseteq)$ are posets. Then, a pair (f, f^{-1}) of maps is a Galois connection between them, because $f(A) \subseteq B$ if and only if $A \subseteq f^{-1}(B)$ for all $A \in 2^X, B \in 2^Y$.

The notion of Galois connections are generalized to the case of L-fuzzy sets for complete residuated lattices L by Bêlohlávek (1999), where L^X is used instead of 2^X. A pair (f, g) of maps is called a *fuzzy Galois connection* between L-fuzzy posets (L^X, Sub_X) and (L^Y, Sub_Y) for maps $f : L^X \to L^Y$ and $g : L^Y \to L^X$, if f, g satisfies the conditions: For all $A, A_1, A_2 \in L^X,\ B, B_1, B_2 \in L^Y$,

(BFG1) $\mathrm{Sub}_X(A_1, A_2) \leq \mathrm{Sub}_Y(f(A_1), f(A_2))$ and
$\mathrm{Sub}_Y(B_1, B_2) \leq \mathrm{Sub}_X(g(B_1), g(B_2))$;

(BFG2) $A \leq gf(A),\ B \leq fg(B)$,

where Sub_X is defined by

$$\mathrm{Sub}_X(A_1, A_2) = \bigwedge_{x \in X} (A_1(x) \to A_2(x)).$$

Sub_Y is similarly defined.

It was also proved that fuzzy Galois connections are in one-to-one correspondence with binary fuzzy relations.

Since a map $\mathrm{Sub}_X : L^X \times L^X \to L$ satisfies the following conditions

(Sub1) $\mathrm{Sub}_X(A, A) = 1 \quad (\forall A \in L^X)$;

(Sub2) $\mathrm{Sub}_X(A_1, A_2) = \mathrm{Sub}_X(A_2, A_1) = 1$ implies $A_1 = A_2$ for all $A_1, A_2 \in L^X$;

(Sub3) $\mathrm{Sub}_X(A_1, A_2) \odot \mathrm{Sub}_X(A_2, A_3) \leq \mathrm{Sub}_X(A_1, A_3)$ for all $A_1, A_2, A_3 \in L^X$,

it can be considered as a fuzzy partial order on the L-fuzzy sets. Yao and Lu (2009) considered the notion L-fuzzy posets as the generalization of posets and proved some fundamental properties about L-fuzzy posets and fuzzy Galois connections.

In this talk, we give operator-based proofs to their results, which make such results can be extended easily to more general cases. Moreover, we show that a new Galois connection arises between L-fuzzy posets.