Compactly Generated de Morgan Lattices

Jan Paseka
and
Zdenka Riečanová

Department of Mathematics and Statistics
Masaryk University Brno

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Outline

1. Introduction
2. Compactly generated Archimedean lattice effect algebras

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Introduction I

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In the Nineties, the Slovak school of quantum structures generalized the concept of MV-algebra with the concept of **D-poset** or equivalently with the concept of **effect algebra**.

Any of the above structures is equipped with a duality operation hence it is a de Morgan poset.

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Introduction II

The aim of this lecture is to look at various important topologies on compactly generated de Morgan lattices.

Basic definitions – de Morgan posets

Definition

A structure $(E \leq,')$ is called a *de Morgan poset* if $(E, \leq)$ is a poset and $'$ is a unary operation with properties:
(i) $a \leq b \Rightarrow b' \leq a'$
(ii) $a = a''$.

In a de Morgan poset we have $a \leq b$ iff $b' \leq a'$, because $a \leq b \Rightarrow b' \leq a' \Rightarrow a'' \leq b'' \Rightarrow a \leq b$.

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Definition - compactly generated lattice

Definition

(1) An element a of a lattice L is called *compact* iff, for any $D \subseteq L$, $a \leq \bigvee D$ implies $a \leq \bigvee F$ for some finite $F \subseteq D$.

(2) A lattice L is called *compactly generated* iff every element of L is a join of compact elements.

Nets

Definition

A *net* $(a_\alpha)_{\alpha \in \mathscr{E}}$ is a set of elements which have indices from a directed set of indices $\mathscr{E}$.

Definition

A net $(a_\alpha)_{\alpha \in \mathscr{E}}$ of elements of the poset P is *increasingly directed* if $a_\alpha \leq a_\beta$ for all $\alpha, \beta \in \mathscr{E}$ such that $\alpha \leq \beta$ and then we write $a_\alpha \uparrow$. If moreover $a = \bigvee\{a_\alpha \mid \alpha \in \mathscr{E}\}$ we write $a_\alpha \uparrow a$ and we called such a net *increasing to* a. The meaning of $a_\alpha \downarrow$ and $a_\alpha \downarrow a$ is dual (*decreasingly directed* or *filtered*).

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(o)-convergence I

Zdenka Riečanová, Order-topological lattice effect algebras, Contributions to General Algebra 15, Proceedings of the Klagenfurt Workshop 2003 on General Algebra, Klagenfurt, Austria, June 19-22,2003, pp.151-160.

Definition

We say that a net $(x_\alpha)_{\alpha\in\mathscr{E}}$ of elements of a poset P ($\mathscr{E}$ is a directed set) *order converges* to a point $x \in P$ if there exist nets $(u_\alpha)_{\alpha\in\mathscr{E}}, (v_\alpha)_{\alpha\in\mathscr{E}} \subseteq P$ such that $u_\alpha \le x_\alpha \le v_\alpha$ for all α and $(u_\alpha)_{\alpha\in\mathscr{E}}$ is nondecreasing with supremum x, $(v_\alpha)_{\alpha\in\mathscr{E}}$ is nonincreasing with infimum x. We write $u_\alpha \uparrow x$, $v_\alpha \downarrow x$ and $x_\alpha \xrightarrow{(o)} x$.

The finest (biggest) topology on P such that $x_\alpha \xrightarrow{(o)} x$ implies topological convergence is called an *order topology on* P, denoted τ_o.

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(o)-convergence II

Theorem

Assume that $(P, \leq)$ is a complete lattice. Then for $x_\alpha \in P, \alpha \in \mathscr{E}$:

$$x_\alpha \xrightarrow{(o)} x, \alpha \in \mathscr{E} \text{ iff } x = \bigvee_{\beta \in \mathscr{E}} \bigwedge_{\alpha \geq \beta} x_\alpha = \bigwedge_{\beta \in \mathscr{E}} \bigvee_{\alpha \geq \beta} x_\alpha .$$

Theorem

Let $(P, \leq)$ be a poset and $F \subseteq P$. Then F is τ_0-closed iff for every net $(x_\alpha)_{\alpha \in \mathscr{E}}$ of elements of P:

(CS) $(x_\alpha \in F, \alpha \in \mathscr{E}, x_\alpha \xrightarrow{(o)} x) \Rightarrow x \in F$.

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Properties of order topology

Theorem

Let L be a bounded lattice. Then for every $a, b \in L$ with $a \leq b$ the interval $[a, b]$ is τ_0-closed.

Lemma

Let E be a de Morgan poset. Then $a_\alpha \uparrow a$ iff $a'_\alpha \downarrow a'$ and $a_\alpha \xrightarrow{(o)} a$ iff $a'_\alpha \xrightarrow{(o)} a'$.

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Let E *be a de Morgan poset. Then* $a_\alpha \uparrow a$ *iff* $a'_\alpha \downarrow a'$ *and* $a_\alpha \xrightarrow{(o)} a$ *iff* $a'_\alpha \xrightarrow{(o)} a'$.

(o)-continuity

Definition

A lattice L is called (o)*-continuous* if, for any net $(x_\alpha)_{\alpha\in\mathscr{E}}$ and any $x, y \in L$, $x_\alpha \uparrow x$ implies $x_\alpha \wedge y \uparrow x \wedge y$.

If L is a (o)-continuous de Morgan lattice then for any nets $(x_\alpha)_{\alpha\in\mathscr{E}}$, $(y_\alpha)_{\alpha\in\mathscr{E}}$ and any $x, y \in L$, $x_\alpha \uparrow x$, $y_\alpha \uparrow y$ implies $x_\alpha \vee y_\alpha \uparrow x \vee y$ and $x_\alpha \wedge y_\alpha \uparrow x \wedge y$.

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order-topological lattice

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Join-dense and meet-dense subsets

Definition

A subset $\mathscr{U}$ of a lattice L is *join-dense* if for any two elements $x, z \in L$ with $x \not\leq z$, there is some $u \in \mathscr{U}$ with $u \leq x$ but $u \not\leq z$. Thus $\mathscr{U}$ is join-dense in L iff each element of L is a join of elements from $\mathscr{U}$. Meet-density is defined dually.

A lattice L is compactly generated iff the set of all compact elements is join-dense in L.

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A lattice L is compactly generated iff the set of all compact elements is join-dense in L.

Uniformity and compactly generated lattices I

Let L be a lattice such that there exists $\mathscr{U}, \mathscr{V} \subseteq L$ such that for every $x \in L$ we have that

$$x = \bigvee\{u \in \mathscr{U} \mid u \leq x\} = \bigwedge\{v \in \mathscr{V} \mid x \leq v\}.$$

Consider the function family $\psi = \{f_u \mid u \in L, u \in \mathscr{U}\} \cup \{g_v \mid v \in L, v \in \mathscr{V}\}$, where $f_u, g_v : L \to \{0,1\}$ are defined by putting

$$f_u(x) = \begin{cases} 1 & \text{iff} \quad u \leq x \\ 0 & \text{iff} \quad u \not\leq x \end{cases} \quad \text{and} \quad g_v(y) = \begin{cases} 1 & \text{iff} \quad x \leq v \\ 0 & \text{iff} \quad x \not\leq v \end{cases}$$

for all $x, y \in L$.

Further, consider the family of pseudometrics on L:
$\Sigma_\psi = \{\rho_u \mid u \in \mathscr{U}\} \cup \{\pi_v \mid v \in \mathscr{V}\}$, where $\rho_u(a,b) = |f_u(a) - f_u(b)|$ and $\pi_v(a,b) = |g_v(a) - g_v(b)|$ for all $a, b \in L$.

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Uniformity and compactly generated lattices II

Let us denote by $\mathscr{U}_\psi$ the uniformity on L induced by the family of pseudometrics Σ_ψ. Further denote by τ_ψ the topology compatible with the uniformity $\mathscr{U}_\psi$.

Then for every net $(x_\alpha)_{\alpha\in\mathscr{E}}$ of elements of L

$$x_\alpha \xrightarrow{\tau_\psi} x \text{ iff } f_u(x_\alpha) \to f_u(x) \text{ and } g_v(x_\alpha) \to g_v(x)$$

for all $u, v \in L, u \in \mathscr{U}, v \in \mathscr{V}$.

This implies, since f_u and g_v is a separating family of functions, that the topology τ_ψ is Hausdorff. Moreover, the intervals $[u,v] = [u,1] \cap [0,v] = f_u^{-1}(\{1\}) \cap g_v^{-1}(\{1\})$ are clopen sets in τ_ψ. Hence any interval $[\bigvee_{i=1}^n u_i, \bigwedge_{i=1}^n v] = \bigcap_{i=1}^n [u_i, v_i]$, $u_i \in \mathscr{U}, v_i \in \mathscr{V}$ is clopen in τ_ψ.

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Compactly generated de Morgan lattices I

Lemma

Let L be a de Morgan lattice such that there exists $\mathscr{U}, \mathscr{V} \subseteq L$, $\mathscr{U}$ directed and join-dense in L and $\mathscr{V}$ filtered and meet-dense in L. Then $\tau_o \subseteq \tau_\psi$.

Theorem

Let L be a de Morgan lattice such that there exists $\mathscr{U}, \mathscr{V} \subseteq L$, $\mathscr{U}$ directed and join-dense in L and $\mathscr{V}$ filtered and meet-dense in L. Then the following conditions are equivalent:

1. $\tau_o = \tau_\psi$.
2. *Elements of $\mathscr{U}$ are compact and elements of $\mathscr{V}$ are cocompact. Hence L is compactly generated by $\mathscr{U}$.*

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Moreover (1) or (2) implies the condition (3).

3. *L is order topological.*

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