ZKOUŠKOVÁ PÍSEMKA Z MATEMATICKÉ ANALÝZY 2, LS 2021-22

PÍSEMKA ČÍSLO 4, VERZE 28.6. 2022

(1)(16 bodů) Vyšetřete konvergenci řady

$$\sum_{n=1}^{\infty} \left(\sqrt[3]{1+\frac{3}{n}} - \sqrt[4]{1+\frac{4}{n}} \right) \left(\frac{1}{n} - \sin\frac{1}{n} \right) n^a$$

v závislosti na parametru $a \in \mathbb{R}$.

(2)(16 bodů) Ukažte, že existuje Riemannův integrál funkce

$$f(x) = \frac{1}{(2+\sin x)(3+\cos x)}, \quad x \in [0, 4\pi],$$

na intervalu $[0, 4\pi]$, a spočtěte jej.

(3)(16 bodů) Nalezněte všechna maximální řešení diferenciální rovnice

$$x^3 y' - xy = 1\ .$$

(4)(12 bodů) Dokažte následující tvrzení: Nechť f je spojitá reálná funkce na $[1, \infty)$ taková, že konverguje Newtonův integrál $\int_1^\infty f$. Pak

$$\sup\left\{ \left| \int_1^x f \right| ; x \in [1, \infty) \right\} < \infty.$$

① $$\sum_{n=1}^{\infty} \underbrace{\left(\sqrt[3]{1+\frac{3}{n}} - \sqrt[4]{1+\frac{4}{n}}\right)}_{a_n} \underbrace{\left(\frac{1}{n} - \sin\frac{1}{n}\right)}_{b_n} n^a$$

- $$\sqrt[3]{1+3x} - \sqrt[4]{1+4x} = 1 + \frac{1}{3}(3x) + \binom{1/3}{2}(3x)^2 - 1 - \frac{1}{4}(4x) - \binom{1/4}{2}(4x)^2 + \varphi(x)$$

$$= x^2\left(9\binom{1/3}{2} - \binom{1/4}{2}16\right) + \varphi(x) \quad , \quad \frac{\varphi(x)}{x^2} \xrightarrow[x\to 0]{} 0$$

$$= \frac{1}{2}x^2 + \varphi(x)$$

- $a_n = \frac{1}{2}\frac{1}{n^2} + \tilde{\varphi}(n)$, tako $\frac{\tilde{\varphi}(n)}{1/n^2} \xrightarrow[n\to\infty]{} 0$

- $x - \sin x = x - \left(x - \frac{x^3}{3!} + \cancel{o}\psi(x^3)\right) = \frac{x^3}{6} + \psi(x)$, $\frac{\psi(x)}{x^3} \xrightarrow[x\to 0]{} 0$

- $b_n = \frac{1}{6}\frac{1}{n^3} + \tilde{\tilde{\varphi}}(n)$, tako $\frac{\tilde{\tilde{\varphi}}(n)}{1/n^3} \xrightarrow[n\to\infty]{} 0$

- $$a_n b_n n^a = \left(\frac{1}{2n^2} + \tilde{\varphi}(n)\right)\left(\frac{1}{6n^3} + \tilde{\tilde{\varphi}}(n)\right)n^a =$$

$$= \frac{1}{12}\frac{1}{n^{5-a}} + \frac{n^a}{2n^2}\tilde{\tilde{\varphi}}(n) + \frac{n^a}{6n^3}\tilde{\varphi}(n) + \tilde{\varphi}(n)\tilde{\tilde{\varphi}}(n)n^a$$

$$\left[\ \frac{\frac{n^a}{2n^2}\tilde{\tilde{\varphi}}(n)}{1/n^{5-a}} = \frac{1}{2}n^3\tilde{\tilde{\varphi}}(n) = \frac{1}{2}\frac{\tilde{\tilde{\varphi}}(n)}{1/n^3} \xrightarrow[n\to\infty]{} 0\right.$$

$$\frac{\frac{n^a}{6n^3}\tilde{\varphi}(n)}{1/n^{5-a}} = \frac{1}{6}\frac{\tilde{\varphi}(n)}{1/n^2} \xrightarrow[n\to\infty]{} 0$$

$$\left.\frac{\tilde{\varphi}(n)\tilde{\tilde{\varphi}}(n)n^a}{1/n^{5-a}} = \frac{\tilde{\varphi}(n)}{1/n^2}\,\frac{\tilde{\tilde{\varphi}}(n)}{1/n^3} \xrightarrow[n\to\infty]{} 0\ \right]$$

- $a_n b_n n^a = \frac{1}{12}\frac{1}{n^{5-a}} + \psi(n) = \frac{1}{n^{5-a}}\left(\frac{1}{12} + \frac{\psi(n)}{1/n^{5-a}}\right) > 0$ od nekog n_0.

$\frac{\psi(n)}{1/n^{5-a}} \xrightarrow[n\to\infty]{} 0$

- uspoređujemo s $\sum \frac{1}{n^{5-a}}$, pa $\frac{a_n b_n n^a}{1/n^{5-a}} = \frac{1}{12} + \frac{\psi(n)}{1/n^{5-a}} \to \frac{1}{12} \in (0,\infty)$

- Naš red konvergira $\Leftrightarrow$ $\sum \frac{1}{n^{5-a}}$ konvergira $\Leftrightarrow$ $5-a > 1 \Leftrightarrow 4 > a$

- $\Gamma - \Gamma$ .. 5
- síla .. 3
- odhad σ .. 4
- zářič .. 4

② $\cdot$ f spojitá na $\langle 0, 4\pi\rangle$, tedy R-integrovatelná a $(R)\int_0^{4\pi} f = (N)\int_0^{4\pi} f$

$$\cdot \int_0^{4\pi} f = \int_{-\pi}^{3\pi} f = \int_{-\pi}^{\pi} f + \int_{\pi}^{3\pi} f = 2\int_{-\pi}^{\pi} f \quad \text{z periodicity}$$

$$\cdot \int_{-\pi}^{\pi} f = \int_{-\pi}^{\pi} \frac{1}{(2+\sin x)} \frac{1}{(3+\cos x)}\, dx = \int_{-\infty}^{\infty} \frac{1}{\left(2+\frac{2t}{1+t^2}\right)\left(3+\frac{1-t^2}{1+t^2}\right)} \frac{2}{1+t^2}\, dt$$

$$\operatorname{tg}\frac{x}{2} = t \qquad \sin x = \frac{2t}{1+t^2}$$

$$dx = \frac{2}{1+t^2}\, dt \qquad \cos x = \frac{1-t^2}{1+t^2}$$

$$= \int_{-\infty}^{\infty} \frac{\cancel{2}}{\frac{\cancel{2}(t^2+t+1)}{1+t^2} \cdot \frac{2(2+t^2)}{\cancel{1+t^2}} \cancel{(1+t^2)}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{t^2+1}{(t^2+t+1)(t^2+2)} =$$

$$\left[\frac{t^2+1}{(t^2+t+1)(t^2+2)} = \frac{At+B}{t^2+2} + \frac{Ct+D}{t^2+t+1} \right.$$

$$t^2+1 = (At+B)(t^2+t+1) + (Ct+D)(t^2+2) = t^3(A+C) + t^2(A+B+D) + t(A+B+2C) + (B+2D)$$

$$A+C = 0 \to C = -A \qquad\qquad 2A+D = 1,\quad A+2D = 1 \Rightarrow A = D = 1/3$$

$$A+B+D = 1$$

$$A+B+2C = 0 \to A+B-2A = B-A = 0 \to B = A \qquad\qquad \Rightarrow C = -1/3,\ B = 1/3$$

$$\left. B+2D = 1 \right]$$

$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} \frac{1}{3}\frac{t+1}{t^2+2} + \left(\frac{1}{3}\right)\frac{-t+1}{t^2+t+1}\right) = \frac{1}{6}\left(\int_{-\infty}^{\infty}\left(\frac{t+1}{t^2+2} + \frac{-t+1}{t^2+t+1}\right)\right) =$$

$$\left[\int \frac{t+1}{t^2+2} = \int \frac{t}{t^2+2} + \frac{1}{t^2+2} = \frac{\log(t^2+2)}{2} + \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{t}{\sqrt{2}} \right.$$

$$\int \frac{-t+1}{t^2+t+1} = -\int \frac{t-1}{t^2+t+1} = -\frac{1}{2}\int \frac{2t-2}{t^2+t+1} = -\frac{1}{2}\int \frac{2t+1-3}{t^2+t+1} = -\frac{1}{2}\log(t^2+t+1) + \frac{3}{2}\int \frac{1}{t^2+t+1} =$$

$$\left. = -\frac{1}{2}\log(t^2+t+1) + \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{t+1/2}{\sqrt{\frac{3}{4}}} \right]$$

$$= \frac{1}{6}\left[\frac{1}{2}\log\frac{t^2+2}{t^2+t+1} + \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{t}{\sqrt{2}} + \frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}\left(t+\tfrac{1}{2}\right)\right)\right]_{-\infty}$$

$$= \frac{1}{6}\left[\left(\frac{1}{2}\cdot 0 + \frac{1}{\sqrt{2}}\frac{\pi}{2} + \frac{2}{\sqrt{3}}\frac{\pi}{2}\right) - \left(\frac{1}{2}\cdot 0 + \frac{1}{\sqrt{2}}\left(-\frac{\pi}{2}\right) + \frac{2}{\sqrt{3}}\left(-\frac{\pi}{2}\right)\right)\right]$$

$$= \frac{1}{6}\left[2\cdot\frac{1}{\sqrt{2}}\frac{\pi}{2} + 2\cdot\frac{2}{\sqrt{3}}\frac{\pi}{2}\right] = \frac{1}{6}(\pi)\left(\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{3}}\right).$$

Tedy $\int_0^{4\pi} f = 2\int_{-\pi}^{\pi} f = \frac{1}{3}\pi\left(\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{3}}\right)$

z = u 1

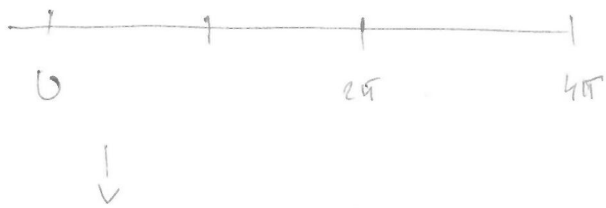
správné ∫ 2

subst 3

parc. zlomky 4

parc ∫ 4

dopočet 2



(3) $x^3y' - xy = 1$

uvažujme rovnici $y' - \frac{y}{x^2} = \frac{1}{x^3}$, $x \in (-\infty, 0), (0, \infty)$

integrační faktor $e^{1/x}$

$$e^{1/x}y' - e^{1/x}\frac{y}{x^2} = e^{1/x}\frac{1}{x^3}$$

$$(y e^{1/x})' = e^{1/x}\frac{1}{x^3} \qquad -\int -e^{1/x}\frac{1}{x^3} \rightsquigarrow -\int e^t t = -te^t + e^t$$

$\frac{1}{x} = t$

$-\frac{1}{x^2}dx = dt$ $\rightsquigarrow \int e^{1/x}\frac{1}{x^3} = e^{1/x}(1 - 1/x)$

$$y e^{1/x} = e^{1/x}(1 - 1/x) + c, \quad c \in \mathbb{R}$$

$$y = 1 - 1/x + c e^{-1/x}, \quad x \in (-\infty, 0), (0, \infty)$$

lepení v 0: $c = 0$, pak $y(x) = 1 - 1/x$ nemá vlastní limitu v 0,

tedy nelze lepit

$c \neq 0$, pak $y(x) = 1 - 1/x + c e^{-1/x}$ $\xrightarrow{x \to 0_+} -\infty$, $\xrightarrow{x \to 0_-} (\operatorname{sgn} c)\infty$

tedy nelze lepit.

Proto maximální řešení jsou:

$$y(x) = 1 - 1/x + c e^{-1/x}, \quad x \in (-\infty, 0), (0, \infty), \quad c \in \mathbb{R}$$

Bodování: dělení x ... 1

i.f 4

integrace pravé strany ... 5

řešení nod. rovnice ... 2

lepení ... 4

4 Ať $\int_1^\infty f$ konverguje. Označme $F(x) = \int_1^x f$, $x \geq 1$, pak F primitivní k f na $(1, \infty)$. Dle předpokladu $\lim_{x \to \infty} F(x) = A \in \mathbb{R}$, dále $\lim_{x \to 1_+} F(x) = F(1) = 0$. Tedy F spojitá na $[1, \infty)$ a v ∞ má vlastní limitu. Proto existuje $z \in (1, \infty)$, že F omezená na (z, ∞). Pak F omezená na $[1, z]$ ze spojitosti, tedy F omezená na $[1, \infty)$. Proto

$$\sup \left\{ \left| \int_1^x f \right| : x \geq 1 \right\} = \sup \{ |F(x)| : x \geq 1 \} < \infty.$$

Bodování ... 12 bodů