a) • $Tx = \sum x_n a_n$ je prvek $(c_0)^* \iff \{a_n\} \in \ell_1$

Dk.: $\Leftarrow$ věta o reprezentaci

"$\Rightarrow$" $T \in (c_0)^* \Rightarrow \exists (b_n) \in \ell_1$, $Tx = \sum x_n b_n \Rightarrow a_n = b_n \Rightarrow (a_n) \in \ell_1$

Tedy $T \in (c_0)^* \iff \sum (\sin 1/n)^\alpha$ konv. $\iff \sum 1/n^\alpha$ konv. $\iff \alpha > 1$

pak $\|T\|_{(c_0)^*} = \sum (\sin 1/n)^\alpha$

• $T \in (\ell_1)^* \iff (a_n) \in \ell_\infty$ (dk. jako výše) $\iff$

$\iff \{(\sin 1/n)^\alpha\} \in \ell_\infty \iff \alpha \geq 0$

• $T \in (\ell_\infty)^* \Rightarrow T1 = \sum (\sin 1/n)^\alpha < \infty \Rightarrow \sum (\sin 1/n)^\alpha < \infty \Rightarrow \alpha > 1$

$\alpha > 1 \Rightarrow |Tx| \leq \sum |x_n| |(\sin 1/n)^\alpha| \leq \|x\|_\infty \sum |\sin 1/n|^\alpha$

Tedy $T \in (\ell_\infty)^* \iff \alpha > 1$

b) • $Tf = f + Kf$, kde $Kf(x) = \int_0^x f$. Jelikož $K \in \mathcal{K}(X)$ (viz Arzela-Ascoli) a $T = I + K$, není T kompaktní. Nicméně $T \in \mathcal{L}(X)$.

• $Tf = \lambda f \Rightarrow f + Kf = \lambda f$

$Kf = (\lambda - 1) f$

• $\lambda = 1 \Rightarrow Kf = 0 \Rightarrow f = 0$

• $\lambda \neq 1 \Rightarrow$ lze derivovat $\Rightarrow f(x) = (\lambda - 1) f'(x) \Rightarrow f(x) = e^{\frac{1}{\lambda-1} x} \cdot c$

$0 = f(0) \Rightarrow c = 0 \Rightarrow f \equiv 0$

$\Rightarrow \sigma_p(T) = \{0\}$

- $(\lambda I - T) f = g$

$\lambda f - f - Kf = g$

$(\lambda - 1) f - Kf = g$

$\swarrow$

$\lambda = 1$, $-Kf = g$

$K(X) \subset \{$ spojité dif fct $\}$

tedy stačí vzít $g \in X$ nedif. a

$g \notin \operatorname{Rng} K$, tj. rovnice nemá řešení

$\lambda \neq 1$, $F(x) := \int_0^x f$, pak máme

$$(\lambda - 1) F' - F = g$$

homogenní rovnice: $(\lambda - 1) F' - F = 0$

$$F = e^{\frac{1}{\lambda-1}x} c$$

variace konstant:

$$F' = \frac{1}{\lambda-1} e^{\frac{1}{\lambda-1}x} c + c' e^{\frac{1}{\lambda-1}x}$$

$$g = (\lambda-1) F' - F = \cancel{e^{\frac{1}{\lambda-1}x} c} + (\lambda-1) c' e^{\frac{1}{\lambda-1}x} - \cancel{c e^{\frac{1}{\lambda-1}x}}$$

$$\Rightarrow c'(x) = \frac{1}{\lambda-1} g(x) e^{-\frac{x}{\lambda-1}}$$

$$c(x) = \int_0^x \frac{1}{\lambda-1} g(t) e^{-\frac{t}{\lambda-1}} \, dt$$

$$\Rightarrow F(x) = \frac{e^{\frac{x}{\lambda-1}}}{\lambda-1} \int_0^x g(t) e^{-\frac{t}{\lambda-1}} \, dt$$

$$f(x) = F'(x) = \frac{1}{\lambda-1} \left[\frac{1}{\lambda-1} e^{\frac{x}{\lambda-1}} \int_0^x g(t) e^{-\frac{t}{\lambda-1}} \, dt + e^{\frac{x}{\lambda-1}} g(x) e^{-\frac{x}{\lambda-1}} \right]$$

$$= \frac{1}{\lambda-1} \left[\frac{1}{\lambda-1} e^{\frac{x}{\lambda-1}} \int_0^x g(t) e^{-\frac{t}{\lambda-1}} \, dt + g(x) \right]$$

$$\Rightarrow \sigma(T) = \{1\}$$