a) · T dobře definovaný, T do c_0 z Riemann-Lebesgueova lemmatu

· $\|Tf\| = \sup_n |Tf(n)| \leq \sup_n \int_0^{2\pi} |f| = \|f\|_{L_1} \Rightarrow \|T\| \leq 1$

· T lineární

· $T^*: \ell_1 \to L_\infty([0, 2\pi])$, ať $y \in \ell_1$

$$T^*y(f) = g(f) = \int_0^{2\pi} f g, \quad \text{kde } g \in L_\infty([0,2\pi])$$

$$\overset{\shortparallel}{y(Tf)} = \sum_{n=1}^{\infty} (Tf)(n)\, y(n) = \sum_{n=1}^{\infty} \left(\int_0^{2\pi} f(t) e^{int}\, dt \right) y(n)$$

$$\overset{(*)}{=} \int_0^{2\pi} f(t) \left(\sum_{n=1}^{\infty} y(n) e^{-int} \right) dt, \quad f \in L_1$$

$$\Rightarrow T^*y = \sum_{n=1}^{\infty} y(n) e^{int}$$

$$(*) \int_0^{2\pi} \sum_{n=1}^{\infty} |f(t)|\, |y(n)|\, |e^{-int}| \leq \int_0^{2\pi} \sum_{n=1}^{\infty} |f(t)|\, |y(n)| \overset{\text{Levi}}{=} \sum |y_n| \int_0^{2\pi} |f|$$

$\leq \|y\|_{\ell_1} \|f\|_{L_1} \Rightarrow$ prohození lze podle Lebesgueovy věty

• volbou $f_n(t) = e^{int}$ máme $\mathrm{Rng}\, T \supset \mathrm{span}\{ e_n : n \in \mathbb{N} \}$ (bázové vektory)

$\Rightarrow \mathrm{Rng}\, T$ je hustý v c_0

b) $Tf(x) = g(x) f(x)$, $f \in L_2^{\mathbb{R}}(\mathbb{R})$, $g(x) = \max\{0, \sin x\}$

· zřejmé: $Tf \in L_2$, T lineární

· $\|Tf\|_{L_2}^2 = \int_{\mathbb{R}} |gf|^2 \leq \|g\|_\infty^2 \|f\|_{L_2}^2 \Rightarrow \|T\| \leq \|g\|_\infty = 1$

· $f_n = \chi_{(\frac{\pi}{2} - \frac{1}{n}, \pi/2)} \Rightarrow \|f_n\|^2 = \int_{\pi/2 - 1/n}^{\pi/2} 1^2 = \frac{1}{n}$

$$\|Tf_n\|^2 = \int_{\pi/2 - 1/n}^{\pi/2} |g|^2 \geq |g(\tfrac{\pi}{2} - \tfrac{1}{n})|^2 \frac{1}{n} = |g(\tfrac{\pi}{2} - \tfrac{1}{n})|^2 \|f_n\|^2$$

$$\Rightarrow \frac{\|Tf_n\|}{\|f_n\|} \geq |g(\tfrac{\pi}{2} - \tfrac{1}{n})| \to g(\tfrac{\pi}{2}) = 1 \Rightarrow \|T\| = 1$$

· $Tf = \lambda f$

$gf = \lambda f \Rightarrow (g-\lambda)f = 0$

$\nearrow \Rightarrow \lambda = 0,\ f = \chi_{(-\pi,0)} \Rightarrow 0 \in \sigma_p(T)$

$\searrow \lambda \neq 0 \Rightarrow g = \lambda$ pouze na množině míry $0 \Rightarrow f \equiv 0$ s.v. $\Rightarrow \lambda \notin \sigma_p(T)$

$\Rightarrow \sigma_p(T) = \{0\}$

·· $(\lambda I - T) f = h$, $h \in L_2(\mathbb{R})$

$\lambda f - gf = h$

$(\lambda - g) f = h$

$\nearrow \lambda \notin [0,1] \Rightarrow f = \frac{h}{g-\lambda} \in L_2$, neboť $(g-\lambda)^{-1}$ omezená, $\lambda \notin \sigma(T)$

$\searrow \lambda \in (0,1) \Rightarrow f = \frac{1}{\lambda - \sin}$ na $(0,\pi)$

$h = \chi_{(0,\pi)}$

nechť $x_0 \in (0,\pi)$, že $\lambda = \sin x_0$. Pak $f(x) = \frac{1}{\sin x_0 - \sin x} =$

$= \frac{1}{\frac{\sin x_0 - \sin x}{x_0 - x}} \cdot \frac{1}{x_0 - x}$, tedy $f \underset{x\to x_0}{\sim} \frac{1}{x_0 - x} \notin L_{2,loc}$, tj. $f \notin L_2(\mathbb{R})$

$\Rightarrow \sigma(T) \supset (0,1) \Rightarrow \sigma(T) = [0,1]$

$\Rightarrow T$ není kompaktní