a) • $a_1 x + a_2 y + a_3 z = 0$, $a_i \in \mathbb{C}$

$\Rightarrow \forall n \in \mathbb{N}: \quad a_1 \cdot \frac{1}{n} + a_2 \cdot \frac{1}{n^2} + a_3 \frac{1}{n^3} = 0$

— // — $\quad a_1 + a_2 \cdot \frac{1}{n} + a_3 \frac{1}{n^2} = 0 \overset{\text{limita}}{\Rightarrow} a_1 = 0$

— // — $\quad a_2 + a_3 \frac{1}{n} = 0 \overset{\text{limita}}{\Rightarrow} a_2 = 0 \Rightarrow a_3 = 0$

• $\tilde{e}_1 := \frac{x}{\|x\|}$, $\quad e_2 = x + cy = x - \frac{\|x\|^2}{\langle y, x\rangle} y$

$0 = \langle e_2, x \rangle = \langle x + cy, x \rangle = \|x\|^2 + c\langle y, x\rangle \Rightarrow c = -\frac{\|x\|^2}{\langle y, x\rangle}$

$$\tilde{e}_2 = \frac{x - \frac{\|x\|^2}{\langle y, x\rangle} y}{\left\| x - \frac{\|x\|^2}{\langle y, x\rangle} y \right\|} \quad \Rightarrow \quad \tilde{e}_1 = \left\{ \frac{1/n}{\sqrt{\sum \frac{1}{n^2}}} \right\}$$

$$\tilde{e}_2 = \frac{1/n - \left(\frac{\sum \frac{1}{n^2}}{\sum \frac{1}{n^3}} \right) \frac{1}{n^2}}{\| \text{ — // — } \|}$$

• $u = \langle z, \tilde{e}_1 \rangle \tilde{e}_1 + \langle z, \tilde{e}_2 \rangle \tilde{e}_2$

b) $(Tx)_n = i^{n-1} x_{n-1}$

• $\|Tx\|^2 = \sum_{n \in \mathbb{Z}} |i^{n-1} x_{n-1}|^2 = \sum_{n \in \mathbb{Z}} |x_{n-1}|^2 = \sum_{k \in \mathbb{Z}} |x_k|^2 = \|x\|^2$

• T lineární $\Rightarrow T \in \mathcal{L}(\ell_2(\mathbb{Z}))$, $\|T\| = 1$

• $Tx = y \Rightarrow i^{n-1} x_{n-1} = y_n \Rightarrow i^k x_k = y_{k+1} \Rightarrow x_k = i^{-k} y_{k+1}$

$n - 1 = k$
$n = k + 1$

$\Rightarrow T^{-1}$ existuje a $\|T^{-1}\| = 1$ jako výše $\Rightarrow \sigma(T) \subseteq \mathbb{T}$

• T^{-1} existuje $\Rightarrow$ T má spektrum

• $Tx = \lambda x$, $\lambda \in \mathbb{T}$

$i^{n-1} x_{n-1} = \lambda x_n \Rightarrow$
$x_1 = \lambda^{-1} i^0 x_0$
$x_2 = \lambda^{-1} i^{2-1} x_1 = \lambda^{-2} i^1 i^0 x_0$
$x_3 = \lambda^{-3} i^2 i^1 i^0 x_0$
$\vdots$
$x_n = \lambda^{-n} i^{n-1} \cdots i^0 x_0$

$\Rightarrow \ |x_n| = |x_0|, \ n \geq 1 \ \Rightarrow \ x_0 = 0 \ \Rightarrow \ x_n = 0, \ n \geq 1$

$x \in \ell_2(\mathbb{Z})$

$\Rightarrow \ i^{-1} x_{-1} = \lambda x_0 \ \Rightarrow \ i^{-2} x_{-2} = \lambda x_{-1} \ \overset{\text{indukcija}}{\Rightarrow} \ x_n = 0, \ n \leq 0$

$x_{-1} = 0 \qquad x_{-2} = 0$

$\Rightarrow \ x = 0 \ \Rightarrow \ \sigma_p(T) = \emptyset$

- $\lambda \in \mathbb{T}$ ddno, $y = \chi_{\{0\}} \ \Rightarrow \ i^{n-1} x_{n-1} - \lambda x_n = y_n, \ n \in \mathbb{Z}$

$n = 0 \ \Rightarrow \ i^{-1} x_{-1} - \lambda x_0 = 1$

$n \geq 1 \ \Rightarrow \ i^{0} x_0 - \lambda x_1 = 0 \ \Rightarrow \ x_1 = \lambda^{-1} i^{0} x_0$

$i^{1} x_1 - \lambda x_2 = 0 \ \Rightarrow \ x_2 = \lambda^{-1} i^{1} x_1 = \lambda^{-2} i^{1} i^{0} x_0$

$\vdots$

$i^{n-1} x_{n-1} - \lambda x_n = 0 \ \Rightarrow \ x_n = \lambda^{-n} i^{(n-1)} \cdots i^{0} x_0$

$\Rightarrow \ |x_n| = |x_0| \ \Rightarrow \ x_0 = 0 \ \Rightarrow \ x_{-1} = i$

$n \geq 1$

$n \leq -1, \quad i^{-2} x_{-2} = \lambda x_{-1} \ \Rightarrow \ x_{-2} = i^{2} \lambda x_{-1}$

$i^{-3} x_{-3} = \lambda x_{-2} \ \Rightarrow \ x_{-3} = i^{3} \lambda x_{-2} = \lambda^{2} i^{3} i^{2} x_{-1}$

$\vdots$

$i^{-n-1} x_{-n-1} = \lambda x_{-n} \ \Rightarrow \ x_{-n-1} = i^{n+1} \lambda x_{-n} = \lambda^{n} i^{n+1} \cdots i^{2} x_{-1}$

$\Rightarrow \ |x_{-n}| = |x_{-1}| = 1, \ n \geq 1 \ \Rightarrow \ x \notin \ell_2(\mathbb{Z})$

$\Rightarrow \ \lambda \in \sigma(T) \ \Rightarrow \ \sigma(T) = \mathbb{T}$