a) $Tf = \int_0^1 f(t^2)\,dt = \int_0^1 f(s)\underbrace{\frac{1}{2\sqrt{s}}}_{g(s)}\,ds = \int_0^1 f(s)\,g(s)\,ds$ (+1)

$t^2 = s$

$t = \sqrt{s}$

$dt = \frac{1}{2\sqrt{s}}\,ds$

(+4). $p \in (1,\infty)$, pak $T \in (L_p)^* \iff g \in L_q$: "$\Leftarrow$" reprezentační věta

"$\Rightarrow$" $T \in (L_p)^* \Rightarrow \exists h \in L_q$: $Tf = \int_0^1 fh$, $f \in L_p$

$\Rightarrow \forall f \in L_p : \int fg = \int fh$

$\Rightarrow \forall A \subset [0,1] : \int_A g = \int_A h \Rightarrow g = h$ s.v. $\Rightarrow g \in L_q$

(+2) tedy $T \in (L_p)^* \iff g \in L_q \iff \int_0^1 (s^{-1/2})^q = \int_0^1 s^{-\frac{q}{2}}$ konverguje $\iff$

$\iff \frac{q}{2} < 1 \iff q < 2 \iff p > 2$

pak $\|T\|_{(L_p)^*} = \|g\|_{L_q} = \left(\int_0^1 \frac{1}{(2s^{1/2})^q}\right)^{1/q}$

$p = \infty$: $\left|\int_0^1 fg\right| \le \|f\|_\infty \int_0^1 |g| \Rightarrow \|T\| \le \int_0^1 |g|$

(+1)

$f \equiv 1 \Rightarrow Tf = \int_0^1 |g| \Rightarrow \|T\| = \int_0^1 \frac{1}{|2\sqrt{s}|}\,ds$

b) $Tf = if + f_1 \varphi(f)$, kde $f_1(t) = t^2$, $\varphi_1(f) = \int_{-1}^{1} f$

· $T_1 f = \varphi(f) f_1$ je prvkem $F(X)$ a $T = iI + T_1 \Rightarrow T \in C(X)$

· T není kpktní: kdyby byl ano, $\frac{T - T_1}{i} = I$ je kpktní, spor

· $\lambda f = Tf = if + \varphi(f) f_1$

$(\lambda - i) f = \varphi(f) f_1 \Rightarrow \lambda = i$, $f \in \ker \varphi$ je vlastní vektor pro i

$\lambda \neq i$: $f = c f_1 \Rightarrow (\lambda - i) c f_1 = \underbrace{\varphi(c f_1)}_{= c \int_{-1}^{1} t^2} f_1 = c \frac{2}{3} f_1$

$\Rightarrow c f_1 (\lambda - i - \frac{2}{3}) = 0 \Rightarrow$ $c = 0 \rightarrow f = 0$; $\lambda = i + \frac{2}{3}$, $c = 1 \Rightarrow f = f_1$ je vl. vektor pro $i + \frac{2}{3}$

$\Rightarrow \sigma_p(T) = \{ i, i + \frac{2}{3} \}$

· $\lambda \notin \sigma_p(T)$, $(\lambda I - T) f = g$, $g \in X$

$\lambda f - if - f_1 \varphi(f) = g$

$f(\lambda - i) = g + f_1 \varphi(f) \quad / \varphi$

$\varphi(f)(\lambda - i) = \varphi(g) + \frac{2}{3} \varphi(f)$

$\varphi(f)(\lambda - i - \frac{2}{3}) = \varphi(g) \Rightarrow f = \frac{1}{\lambda - i} \left(g + f_1 \frac{\varphi(g)}{\lambda - i - \frac{2}{3}} \right)$

$\Rightarrow \sigma(T) = \sigma_p(T)$