a) $T: \ell_1 \to \ell_2$ má matici

$$\begin{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} & & \\ & & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} & \\ 0 & & & \ddots \end{pmatrix}$$

Označme $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ jako operátor $A : (\mathbb{C}^2, \|\cdot\|_2) \to (\mathbb{C}^2, \|\cdot\|_2)$.

Nechť $x \in B_{\ell_1}$, pak $|x_i| \leq 1$ pro $\forall i \in \mathbb{N}$. Tedy

$$\|Tx\|_{\ell_2}^2 = \left\| A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_2^2 + \left\| A \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \right\|_2^2 + \dots = \sum_{n=1}^{\infty} \left\| A \begin{pmatrix} x_{2n-1} \\ x_{2n} \end{pmatrix} \right\|_2^2 \leq$$

$$\leq \sum_{n=1}^{\infty} \left(\|A\|^2 \left\| \begin{pmatrix} x_{2n-1} \\ x_{2n} \end{pmatrix} \right\|_2^2 \right) = \|A\|^2 \sum_{n=1}^{\infty} \left(|x_{2n-1}|^2 + |x_{2n}|^2 \right) =$$

$$= \|A\|^2 \sum_{i=1}^{\infty} |x_i|^2 \leq \|A\|^2 \sum_{i=1}^{\infty} |x_i| \leq \|A\|^2$$

$\Rightarrow \cdot \sup_{x \in B_{\ell_1}} \|Tx\|_{\ell_2} \leq \|A\| \Rightarrow \|T\| \leq \|A\|$

- $Tx \in \ell_2$ pro $x \in \ell_1$
- T zjevně lineární

- $T^* : \ell_2 \to \ell_\infty$

$$T^*y(x) = z(x) = \sum_{k=1}^{\infty} x_k z_k$$

$$\overset{\shortparallel}{y(Tx)} = \sum_{n=1}^{\infty} \left((x_{2n-1} - x_{2n}) y_{2n-1} + (x_{2n-1} + x_{2n}) y_{2n} \right) =$$

$$= \sum_{n=1}^{\infty} \left(x_{2n-1} (y_{2n-1} + y_{2n}) + x_{2n} (-y_{2n-1} + y_{2n}) \right)$$

$\Rightarrow T^*y = (y_1 + y_2, -y_1 + y_2, y_3 + y_4, -y_3 + y_4, \dots, y_{2n-1} + y_{2n}, -y_{2n-1} + y_{2n}, \dots)$

- T^* není na, neboť ℓ_∞ neseparabilní a ℓ_2 separabilní

b) $Tf = f_1 \langle f, f_2 \rangle + f_2 \langle f, f_1 \rangle$, kde $f_1 = e^{-s}$
$f_2 = e^{-2s}$

· $f_1, f_2 \in L_2$ => T je spojitý, kon. dimenzionální => $T \in K(X)$.

· spektrum bodové: $0 \in \sigma_p(T)$, neboť stačí vzít $f \in \{f_1, f_2\}^\perp$, což má kodimenzi nejvýše 2

· $\lambda \neq 0$: $Tf = \lambda f$ => $f = a_1 f_1 + a_2 f_2$, $a_1, a_2 \in \mathbb{C}$

=> $f_1 \langle f, f_2 \rangle + f_2 \langle f, f_1 \rangle = \lambda a_1 f_1 + \lambda a_2 f_2$

$\langle f, f_2 \rangle = a_1 \langle f_1, f_2 \rangle + a_2 \langle f_2, f_2 \rangle = a_1 \frac{1}{3} + a_2 \frac{1}{4}$

$\langle f, f_1 \rangle = a_1 \langle f_1, f_1 \rangle + a_2 \langle f_2, f_1 \rangle = a_1 \frac{1}{2} + a_2 \frac{1}{3}$

=> $f_1 \left(a_1 \frac{1}{3} + a_2 \frac{1}{4}\right) + f_2 \left(a_1 \frac{1}{2} + a_2 \frac{1}{3}\right) = f_1(\lambda a_1) + f_2(\lambda a_2)$

f_1, f_2 LN =>
$$\begin{pmatrix} 1/3 & 1/4 \\ 1/2 & 1/3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad (*)$$

=> $\det \begin{pmatrix} 1/3 - \lambda & 1/4 \\ 1/2 & 1/3 - \lambda \end{pmatrix} = (1/3 - \lambda)^2 - \frac{1}{8} = \frac{1}{9} - \frac{2}{3}\lambda - \frac{1}{8} + \lambda^2 =$

$= \lambda^2 - \frac{2}{3}\lambda - \frac{1}{72}$, existují 2 různé kořeny $\lambda_1, \lambda_2 \in \mathbb{R}$, kde

$\lambda_{1,2} = \dfrac{\frac{2}{3} \pm \sqrt{\frac{4}{9} + \frac{4}{72}}}{2}$, pro λ_i existuje $\begin{pmatrix} a_1^i \\ a_2^i \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ řešící $(*)$

pak $a_1^i f_1 + a_2^i f_2$ musí být vlastní vektor pro λ_i

=> $\sigma_p(T) = \{0, \lambda_1, \lambda_2\}$

· T kompaktní => $\sigma(T) = \sigma_p(T)$