a) $T: \ell_1 \to c_0$

$(x_n) \mapsto \left(\sum_1^\infty x_k, \sum_2^\infty x_k, \sum_3^\infty x_k, \dots\right)$

· T zjevně lineární a $\lim (Tx)_n = 0$ z ~~[illegible]~~ B.C. podmínky konvergence

· $\|Tx\| = \sup_n \left|\sum_{k=n}^\infty x_k\right| \le \|x\|_{\ell_1} \Rightarrow \|T\| \le 1$

· $T: \ell_1 \to c_0 \Rightarrow T^*: \ell_1 \to \ell_\infty$

$T^* y(x) = z(x) = \sum_{k=1}^\infty x_k z_k \quad , y \in \ell_1$

$\forall x \in \ell_1$

$y(Tx) = \sum_{n=1}^\infty (Tx)_n y_n = \sum_{n=1}^\infty \left(\sum_{k=n}^\infty x_k\right) y_n = \sum_{\substack{1 \le n < \infty \\ n \le k < \infty}} x_k y_n =$

$= \sum_{k=1}^\infty \sum_{n=1}^k x_k y_n = \sum_{k=1}^\infty x_k \left(\sum_{n=1}^k y_n\right) \Rightarrow (T^* y)_k = \sum_{n=1}^k y_n, \; y \in \ell_1$

F.V.: $\sum_{\substack{1 \le n < \infty \\ n \le k < \infty}} |x_k||y_n| \le \sum_{k=1}^\infty |x_k| \sum_{n=1}^k |y_n| \le \|x\|_{\ell_1} \|y\|_{\ell_1}$

· pokud T izomorfismus do, existuje $C > 0$, ž $\|Tx\|_{c_0} \ge C \|x\|_{\ell_1}$

AE $x^n = (\underbrace{\tfrac{1}{2^n}, -\tfrac{1}{2^n}, \tfrac{1}{2^n}, -\tfrac{1}{2^n}, \dots \tfrac{1}{2^n}, -\tfrac{1}{2^n}}_{2^n}, 0, 0, 0, \dots)$

$\Rightarrow Tx^n = (\underbrace{0, -\tfrac{1}{2^n}, 0, -\tfrac{1}{2^n}, 0, \dots, -\tfrac{1}{2^n}}_{2^n}, 0, 0, \dots)$

$\Rightarrow \|Tx^n\|_{c_0} = \left|\tfrac{1}{2^n}\right|, \; \|x^n\|_{\ell_1} = 1 \Rightarrow \tfrac{1}{2^n} \ge C \cdot 1, \; n \in \mathbb{N}$, spor

b) $T: L_2([0,1]) \to L_2([0,1])$

$$Tf(t) = \int_0^t s^2 f(s)\,ds, \quad f \in L_2([0,1]),\ t \in [0,1]$$

- $f \in L_2 \Rightarrow f \in L_1 \Rightarrow s^2 f(s) \in L_1 \Rightarrow Tf \in AC[0,1] \Rightarrow Tf \in L_2$

- T lineární

- $\|Tf\|_{L_2}^2 = \int_0^1 |Tf(t)|^2 dt = \int_0^1 \left|\int_0^t s^2 f(s)\,ds\right|^2 dt \le$

$\le \int_0^1 \left(\int_0^1 |f(s)|\,ds\right)^2 dt \overset{\text{Hölder}}{\le} \int_0^1 \int_0^1 |f(s)|^2\,ds\,dt = \|f\|_{L_2}^2 \Rightarrow \|T\| \le 1$

- T kptní, $\tilde{T}: L_2([0,1]) \to C([0,1])$

 $f \in B_{L_2} \Rightarrow |\tilde{T}f(t)| \le \int_0^1 |f(s)|\,ds \le \left(\int_0^1 |f|^2\right)^{1/2} \le 1$

 $\Rightarrow T(B_{L_2}) \subset B_{C[0,1]}$

 $f \in B_{L_2},\ 0 \le t_1 < t_2 \le 1 \Rightarrow |\tilde{T}f(t_2) - \tilde{T}f(t_1)| \le \int_{t_1}^{t_2} |s^2 f(s)|\,ds \le \int_{t_1}^{t_2} |f(s)|\,ds \le$

 $\le \left(\int_{t_1}^{t_2} 1^2\right)^{1/2} \int_{t_1}^{t_2} |f|^2 \le \sqrt{t_2 - t_1} \cdot 1$

Arzela-Ascoli $\Rightarrow$ $\tilde{T}: L_2[0,1] \to C[0,1]$ kptní

$T = I \circ \tilde{T}$, kde $I: C[0,1] \to L_2[0,1]$ spojité

[$\|If\|_{L_2}^2 = \int_0^1 |f|^2 \le \|f\|_\infty^2$]

$\Rightarrow$ T kptní

- $\sigma_p(T)$: $\lambda = 0 \Rightarrow \int_0^t s^2 f(s)\,ds = 0$ [$s^2 f(s) \in L_1 \Rightarrow \int_0^t s^2 f(s)\,ds \in AC$ a lze derivovat]

 $\Rightarrow t^2 f(t) = 0$ s.v.

 $\Rightarrow f = 0$ s.v. $\Rightarrow 0 \notin \sigma_p(T)$

 $\lambda \ne 0$: $\int_0^t s^2 f(s)\,ds = \lambda f(t)$ [LS je AC $\Rightarrow$ PS je AC $\Rightarrow$ LS je C^1 $\Rightarrow$ PS je C^1]

 $\Rightarrow t^2 f(t) = \lambda f'(t)$

 $\Rightarrow f(t) = k e^{\frac{t^3}{3\lambda}}$

 poč. podmínka: $f(0) = 0$ $\Big\} \Rightarrow f = 0 \Rightarrow \lambda \notin \sigma_p(T) \Rightarrow \sigma_p(T) = \emptyset$

- $\sigma(T) = \{0\}$