a) $Y = \operatorname{span}\{e^{-t}, e^{-2t}\} \subset L^2((0,\infty))$, $f(t) = e^{-3t}$

$e_1 := e^{-t} \qquad \tilde{e}_1 = e^{-t}$

$e_2 := e^{-2t} \qquad \tilde{e}_2 = ce^{-t} + e^{-2t}$ má splňovat:

$$0 = \langle e_1, \tilde{e}_2 \rangle = \int_0^\infty e^{-t}(ce^{-t} + e^{-2t}) = \int_0^\infty ce^{-2t} + e^{-3t} = \frac{c}{2} + \frac{1}{3} \Rightarrow c = -\frac{2}{3}$$

$$\Rightarrow \tilde{e}_2 = e^{-2t} - \frac{2}{3}e^{-t}$$

$\cdot\ \|\tilde{e}_1\|^2 = \int_0^\infty (e^{-t})^2 = \int_0^\infty e^{-2t} = 1/2$

$\cdot\ \|\tilde{e}_2\|^2 = \int_0^\infty \left(e^{-2t} - \frac{2}{3}e^{-t}\right)^2 = \int_0^\infty \left(\frac{4}{9}e^{-2t} + e^{-4t} - \frac{4}{3}e^{-3t}\right) = \frac{4}{9}\,\frac{1}{2} + \frac{1}{4} - \frac{4}{3}\,\frac{1}{3} =$

$$= \frac{2}{9} + \frac{1}{4} - \frac{4}{9} = \frac{1}{4} - \frac{2}{9} = \frac{9-8}{36} = \frac{1}{36}$$

$\Rightarrow$ ON-báze $Y = \operatorname{span}\left\{\frac{\tilde{e}_1}{\|\tilde{e}_1\|}, \frac{\tilde{e}_2}{\|\tilde{e}_2\|}\right\} = \operatorname{span}\left\{\overset{f_1}{\frac{e^{-t}}{\frac{\sqrt{2}}{2}}}, \overset{f_2}{\frac{e^{-2t} - \frac{2}{3}e^{-t}}{\frac{1}{6}}}\right\}$

•OG projekce $P: X \to Y$ je dána jako $Py = \sum_{i=1}^{2} \langle \varphi, f_i \rangle f_i$

tedy hledané $g \in Y$ je:

$$g = \left(\int_0^\infty \left(e^{-3t}\frac{e^{-t}}{\frac{\sqrt{2}}{2}}\right)\right)\frac{e^{-t}}{\frac{\sqrt{2}}{2}} + 6\left(\int_0^\infty e^{-3t}\left(e^{-2t} - \frac{2}{3}e^{-t}\right)\right)6\left(e^{-2t} - \frac{2}{3}e^{-t}\right)$$

$$= (\sqrt{2})^2 \cdot \frac{1}{4}e^{-t} + 36\left(\frac{1}{5} - \frac{2}{3}\,\frac{1}{4}\right)\left(e^{-2t} - \frac{2}{3}e^{-t}\right)$$

$$= \frac{e^{-t}}{2} + 36 \cdot \frac{1}{30}\left(e^{-2t} - \frac{2}{3}e^{-t}\right) = \frac{e^{-t}}{2} + \frac{6}{5}\left(e^{-2t} - \frac{2}{3}e^{-t}\right)$$

b) $Tx = \left(2x_1 + x_2,\ 3x_1 + 2x_2,\ \frac{x_4}{4},\ \frac{x_5}{5},\ \frac{x_6}{6},\ \frac{x_7}{7}, \dots\right),\ x \in \ell_1 \to X$

- $\|Tx\|_{\ell_1} = |2x_1 + x_2| + |3x_1 + 2x_2| + \left|\frac{x_4}{4}\right| + \left|\frac{x_5}{5}\right| + \dots \le$

  $\le 2|x_1| + 2|x_2| + 3|x_1| + 3|x_2| + |x_4| + |x_5| + |x_6| \le$

  $\le 5|x_1| + 5|x_2| + |x_3| + |x_4| + \dots \le 5\|x\|_{\ell_1} \Rightarrow \|T\| \le 5$

- T zřejmě lineární

- T kpt: $T_n x = \left(2x_1 + x_2,\ 3x_1 + 2x_2,\ \frac{x_4}{4}, \dots, \frac{x_n}{n}, 0, \dots\right),\ n \ge 4$

  pak • $T_n \in L(X)$ (jako výše)
  • Rng T_n konečně dimenzionální $\Bigg\} \Rightarrow T_n \in K(X)$

$n \ge 4$, $\|T - T_n\| = \sup_{x \in B_{\ell_1}} \|(T - T_n)x\| = \sup_{x \in B_{\ell_1}} \left\|\left(0, \dots, 0, \frac{x_{n+1}}{n+1}, \frac{x_{n+2}}{n+2}, \dots\right)\right\| =$

$= \sup_{x \in B_{\ell_1}} \sum_{k=n+1}^{\infty} \left|\frac{x_k}{k}\right| \le \sup_{x \in B_{\ell_1}} \frac{1}{n+1} \sum_{k=1}^{\infty} |x_k| = \frac{1}{n+1} \xrightarrow[n \to \infty]{} 0$

$\Rightarrow T \in K(X)$

- $\sigma_p(T)$: $Tx = \lambda x$: $\left(2x_1 + x_2,\ 3x_1 + 2x_2,\ \frac{x_4}{4},\ \frac{x_5}{5}, \dots\right) = \lambda(x_1, x_2, x_3, \dots)$

  • $\lambda = 0$: $e_3 = (0, 0, 1, 0, \dots) \in \operatorname{Ker} T \Rightarrow 0 \in \sigma_p(T)$

  • $\lambda \ne 0$: $\begin{aligned} 2x_1 + x_2 &= \lambda x_1 \\ 3x_1 + 2x_2 &= \lambda x_2 \end{aligned} \Rightarrow \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow$ hledáme $\sigma\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$; $\det\begin{pmatrix} 2-\lambda & 1 \\ 3 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 3 = 4 - 4\lambda + \lambda^2 - 3 =$

$= \lambda^2 - 4\lambda + 1 \Rightarrow \lambda_{1,2} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

$\Rightarrow$ • ex. $(x_1, x_2) \in \mathbb{C}^2$ nenulový, že $(x_1, x_2) \in \operatorname{Ker}\left(\lambda_1 I - \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\right)$

• ex. $(y_1, y_2) \in \mathbb{C}^2$ – ıı –, že $(y_1, y_2) \in \operatorname{Ker}\left(\lambda_2 I - \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\right)$

$\Rightarrow x = (x_1, x_2, 0, 0, \dots) \in \operatorname{Ker}(\lambda_1 I - T) \Rightarrow \lambda_1, \lambda_2 \in \sigma_p(T)$

$y = (y_1, y_2, 0, 0, \dots) \in \operatorname{Ker}(\lambda_2 I - T)$

$\lambda \notin \{0, \lambda_1, \lambda_2\}$, $Tx = \lambda x \Rightarrow x_1 = x_2 = 0$

$$\frac{x_4}{4} = \lambda x_3 \Rightarrow x_4 = \lambda \cdot 4 \cdot x_3$$

$$\frac{x_5}{5} = \lambda x_4 \Rightarrow x_5 = \lambda 5 \cdot x_4 = \lambda^2 5 \cdot 4\, x_3$$

$$\frac{x_6}{6} = \lambda x_5 \Rightarrow x_6 = \lambda \cdot 6 \cdot x_5 = \lambda^3 6 \cdot 5 \cdot 4\, x_3$$

$$\vdots$$

$$x_n = \lambda^{n-3} \big(n(n-1) \cdots 4 \big) x_3$$

- $x_3 = 0 \Rightarrow x_n = 0$ pra $n \geq 3$
- $x_3 \neq 0 \Rightarrow \lim_{n \to \infty} |x_n| = \lim_{n \to \infty} \frac{n!}{3 \cdot 2} |\lambda|^{n-3} |x_3| = \infty \Rightarrow x \notin \ell_1$

$\Rightarrow \lambda \notin \sigma_p(T) \Rightarrow \sigma_p(T) = \{0, \lambda_1, \lambda_2\}$

- T Ep hi $\Rightarrow \sigma(T) = \{0, \lambda_1, \lambda_2\}$