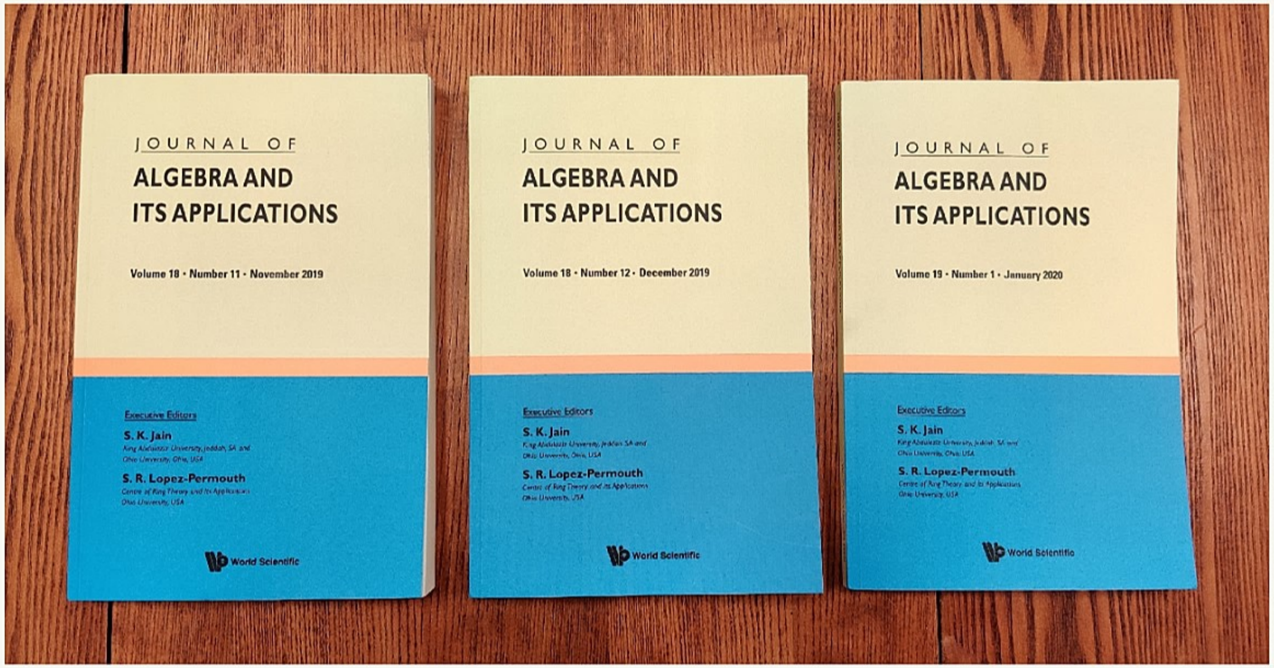
ENTROPY FOR QUANDLES

Filippo Spoggiari
Charles University Prague

FALL SCHOOL
OF ALGEBRA

30 NOV – 3 DIC
2023

Small Seminar Room - Department of Algebra, Charles University Prague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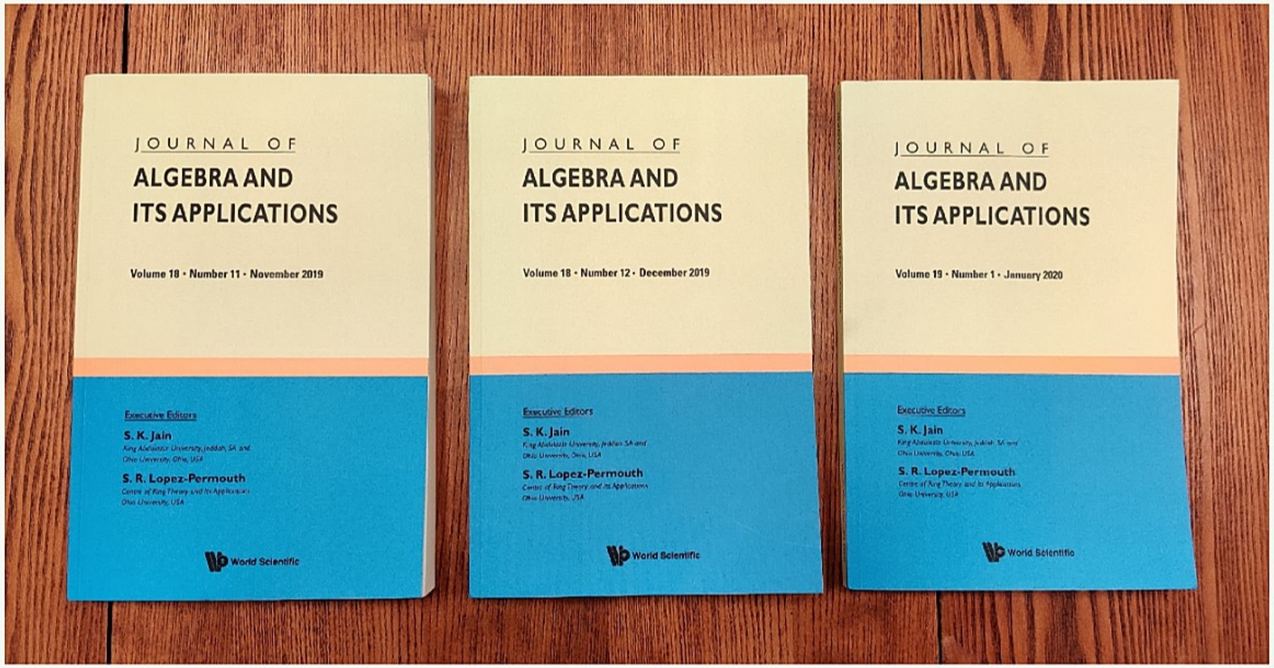
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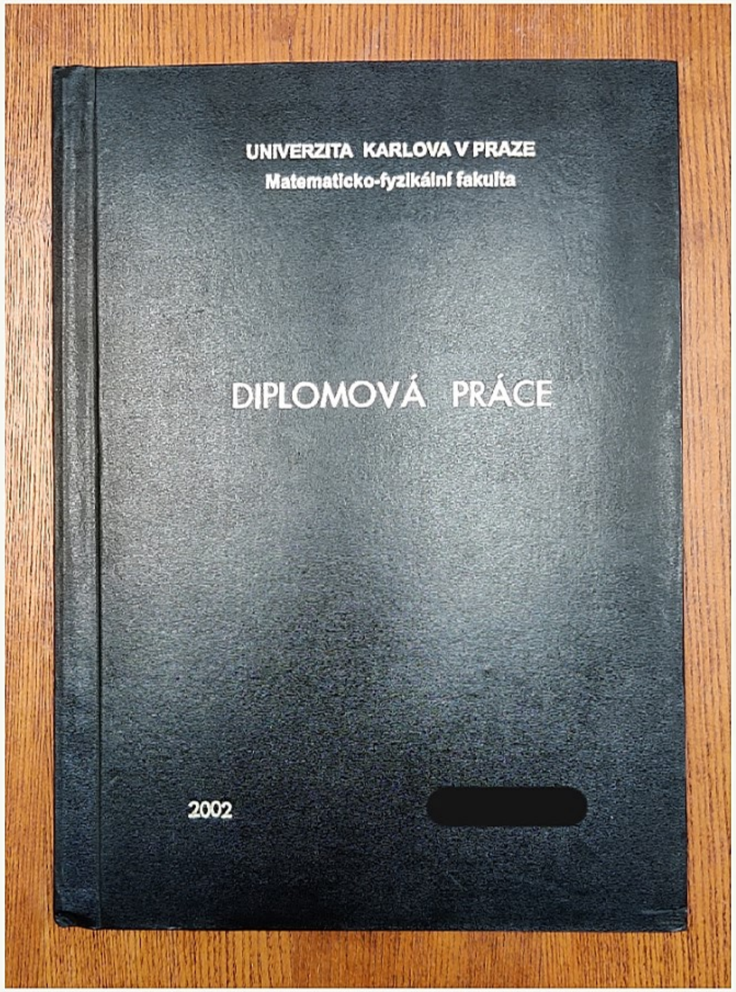
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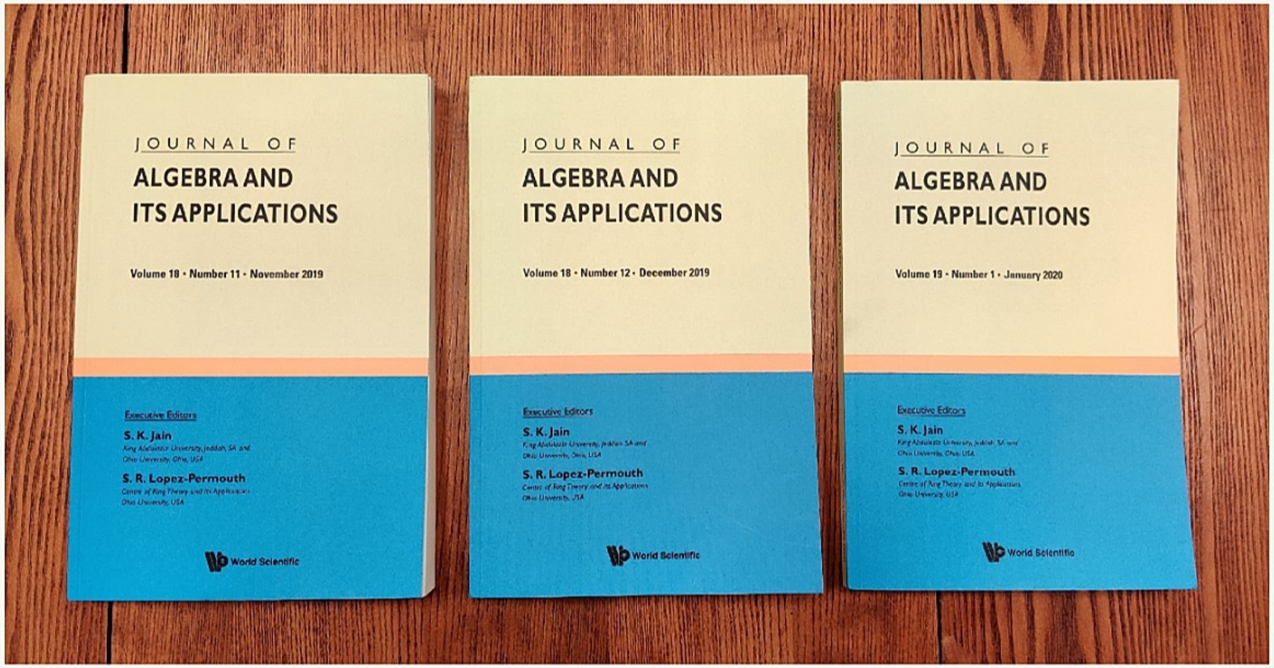
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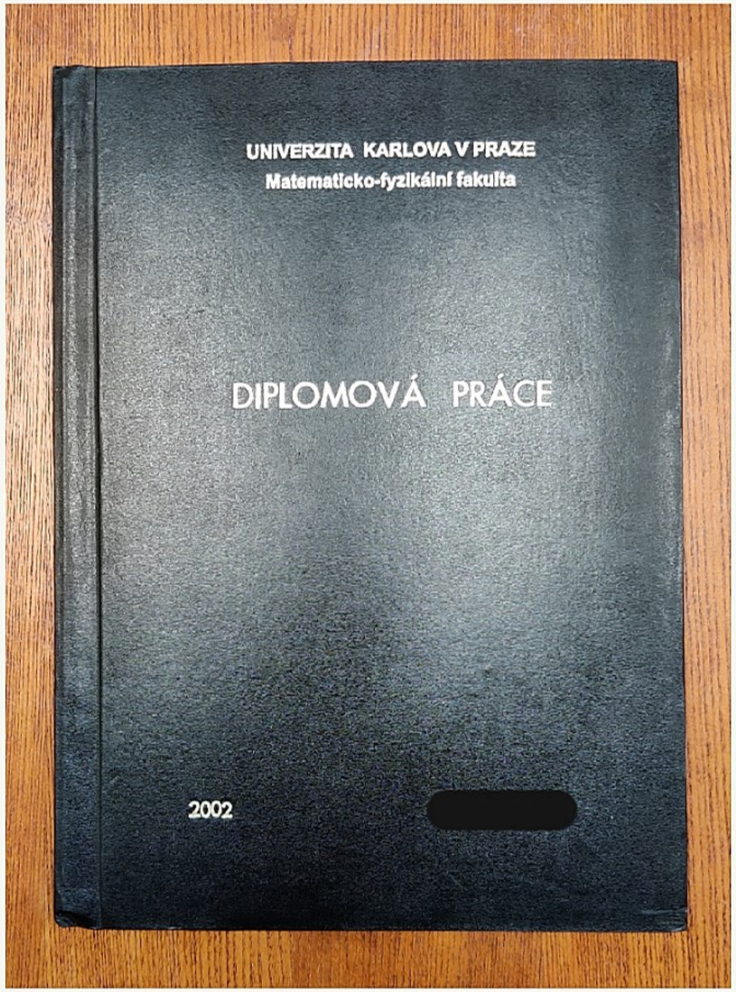
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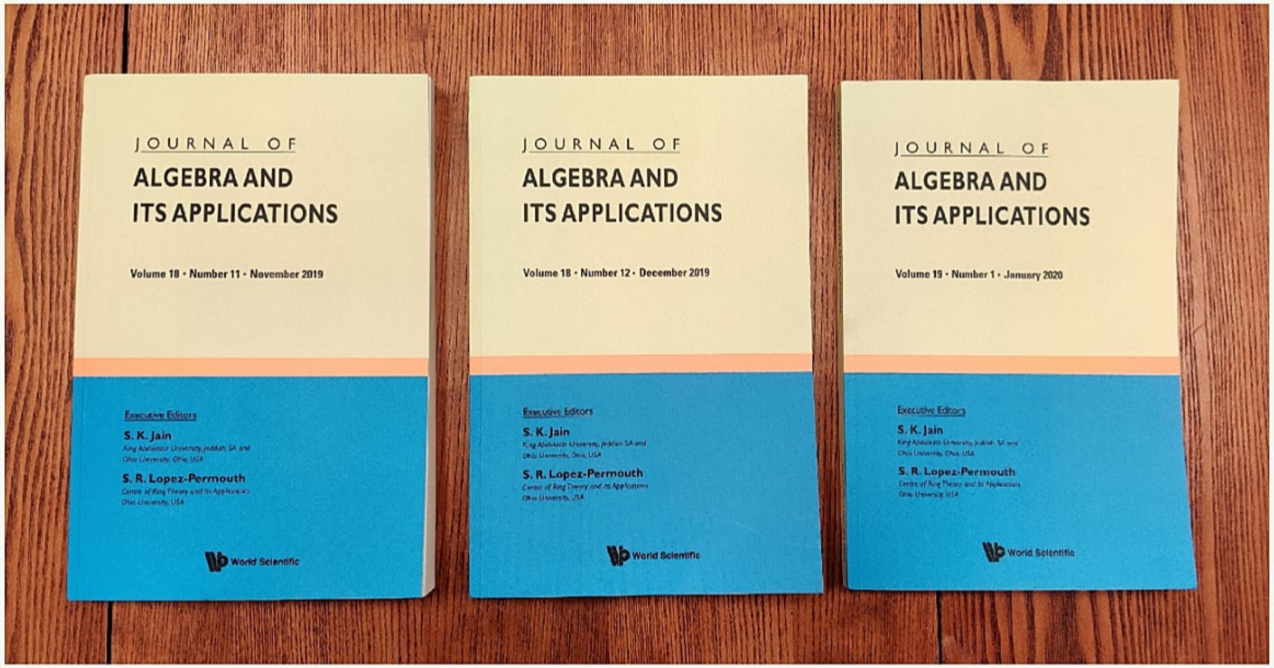
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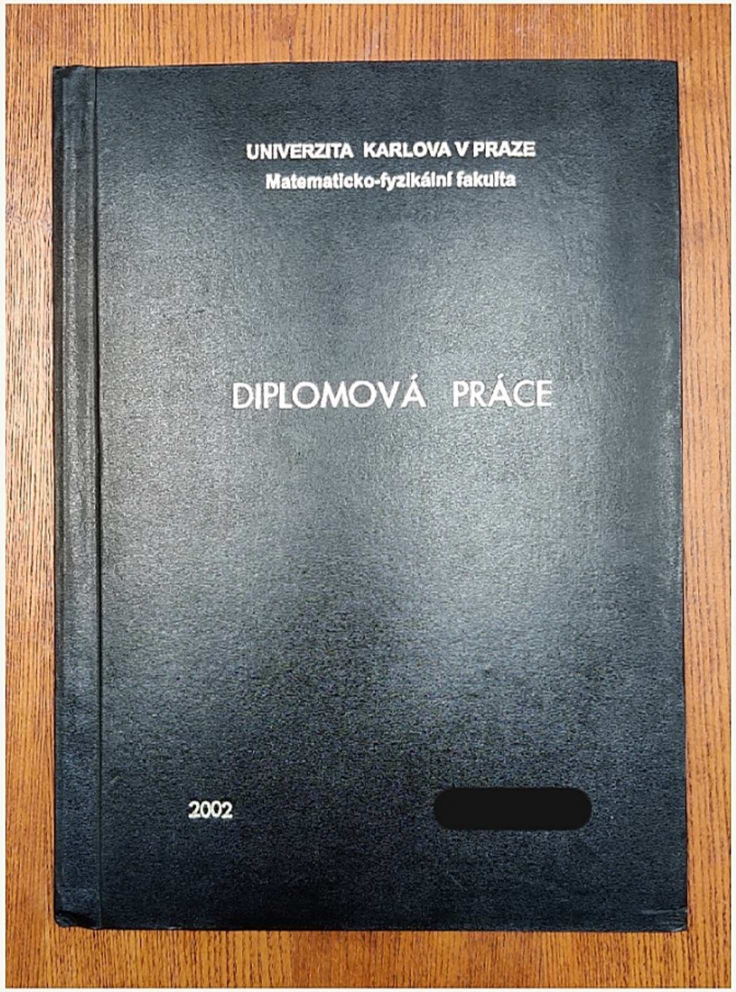
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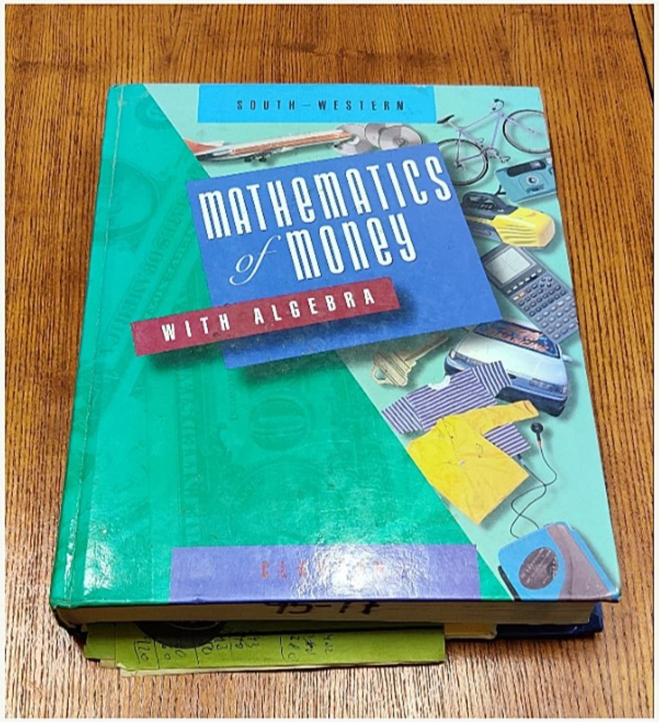
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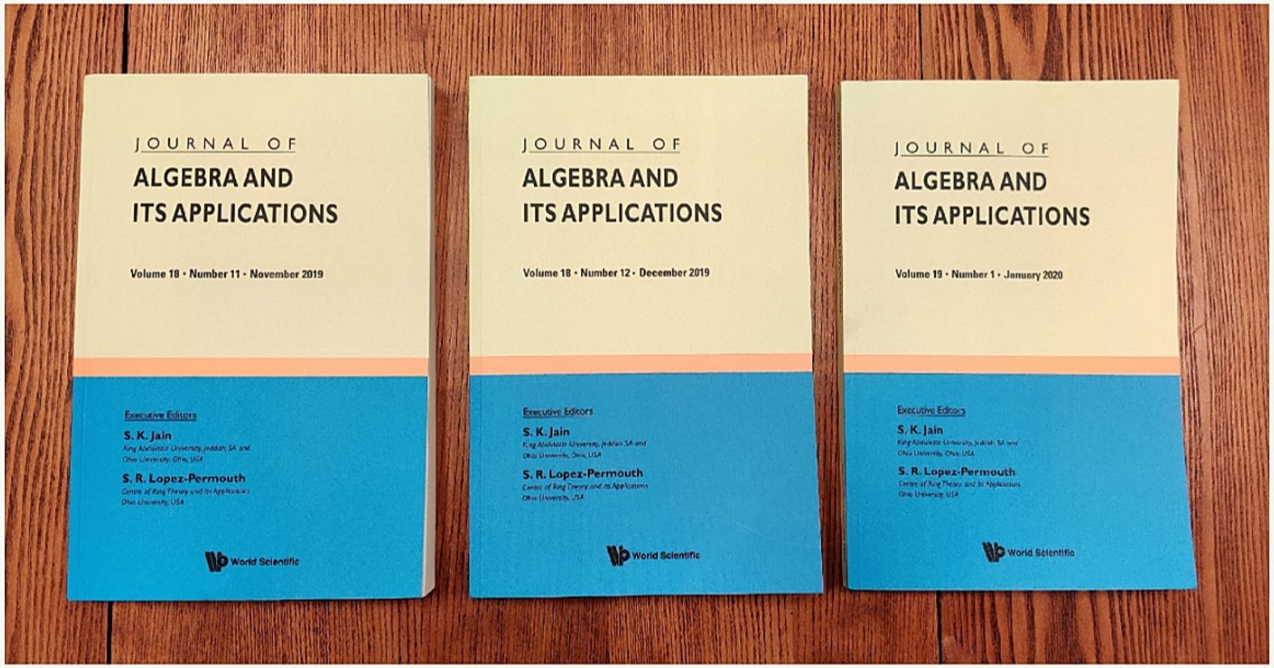
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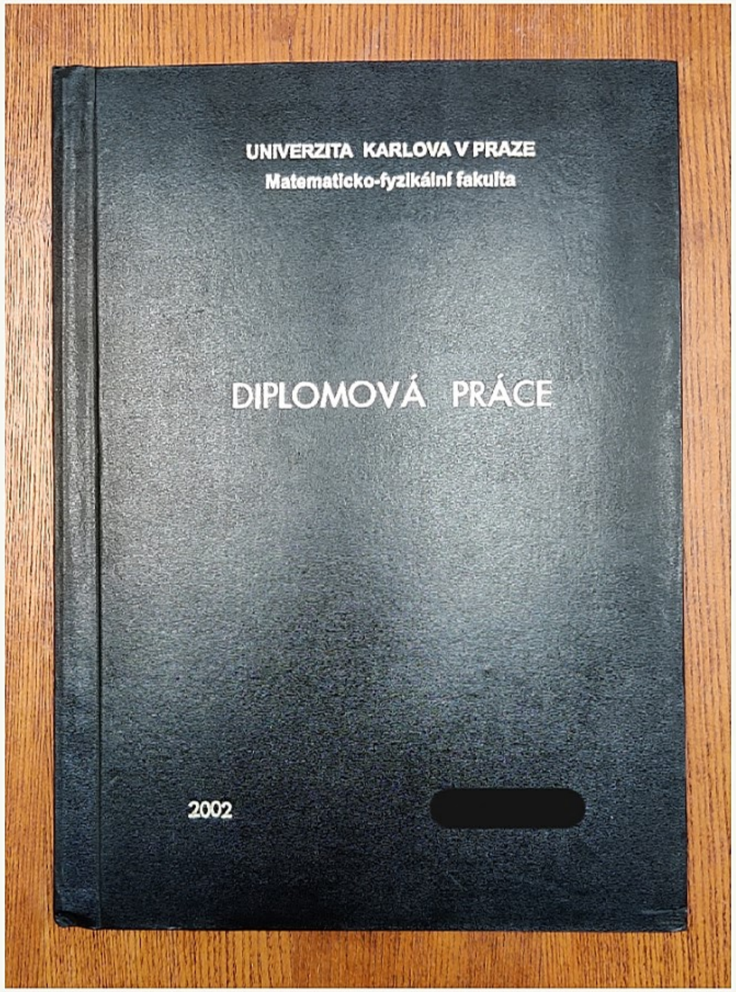
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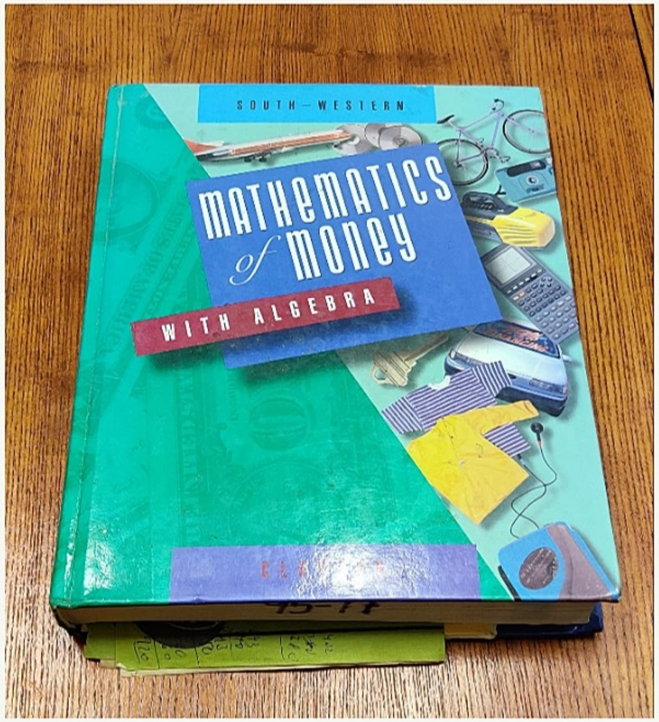
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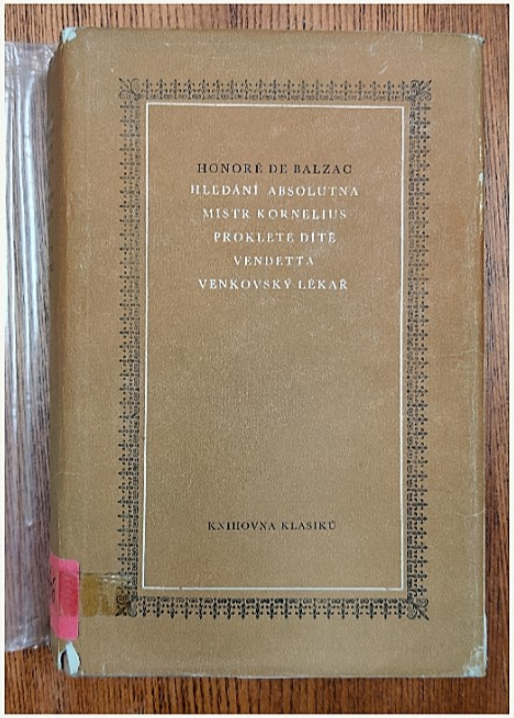
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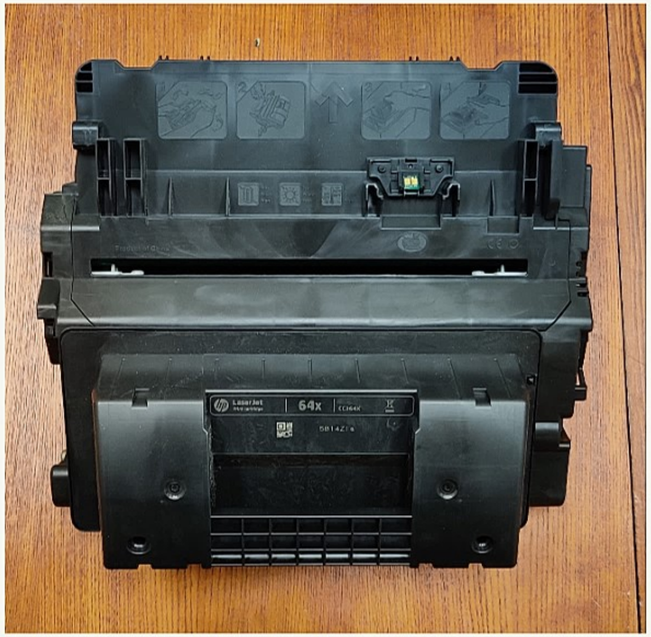
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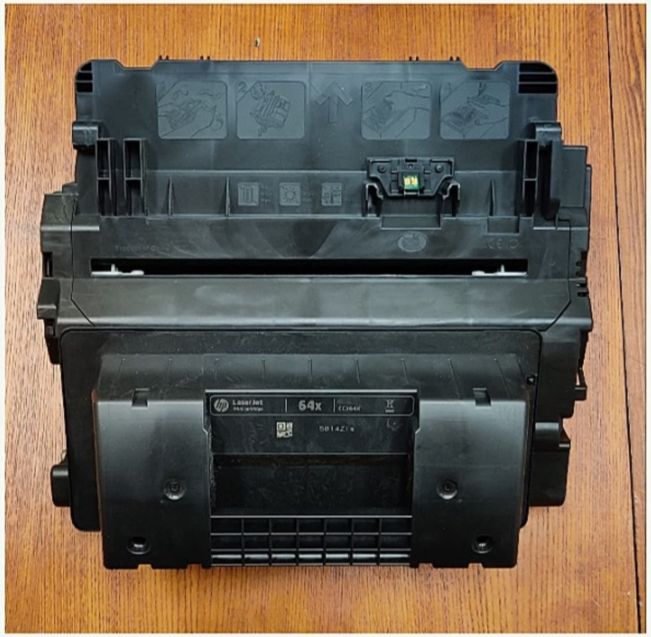


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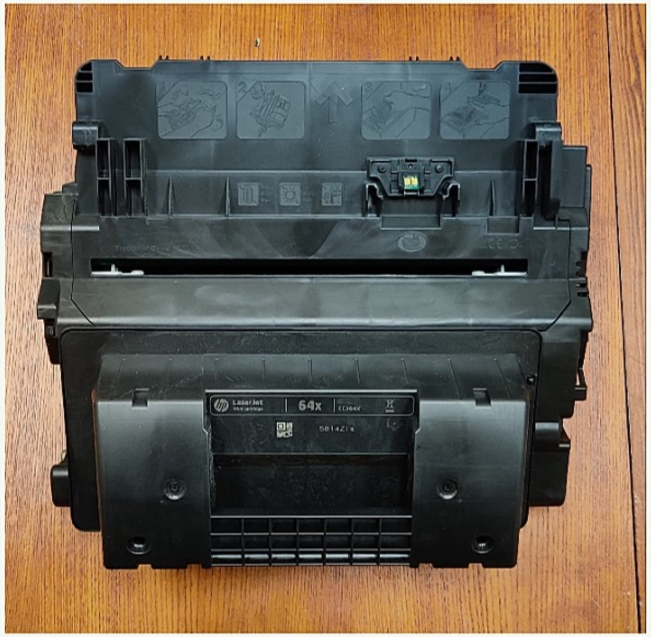
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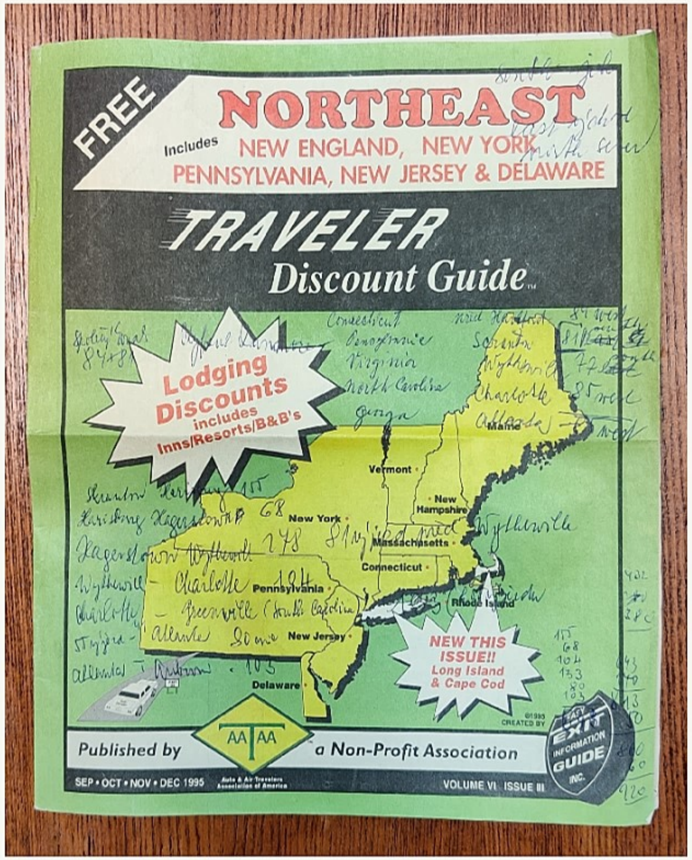
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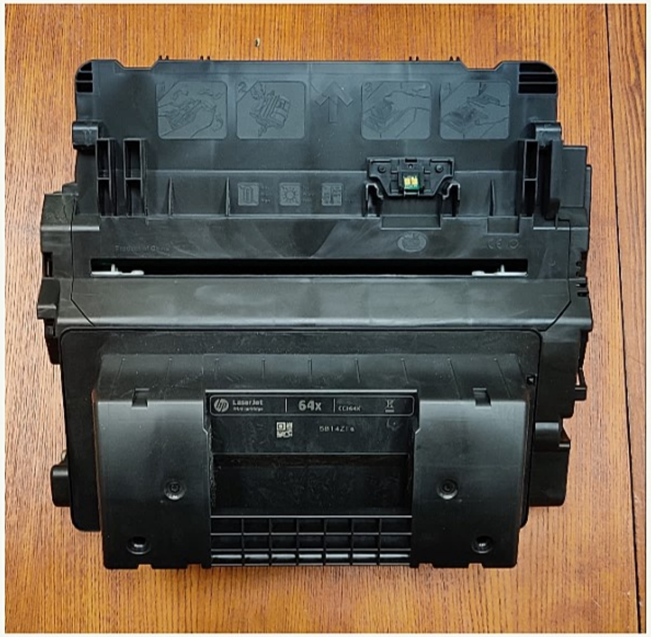
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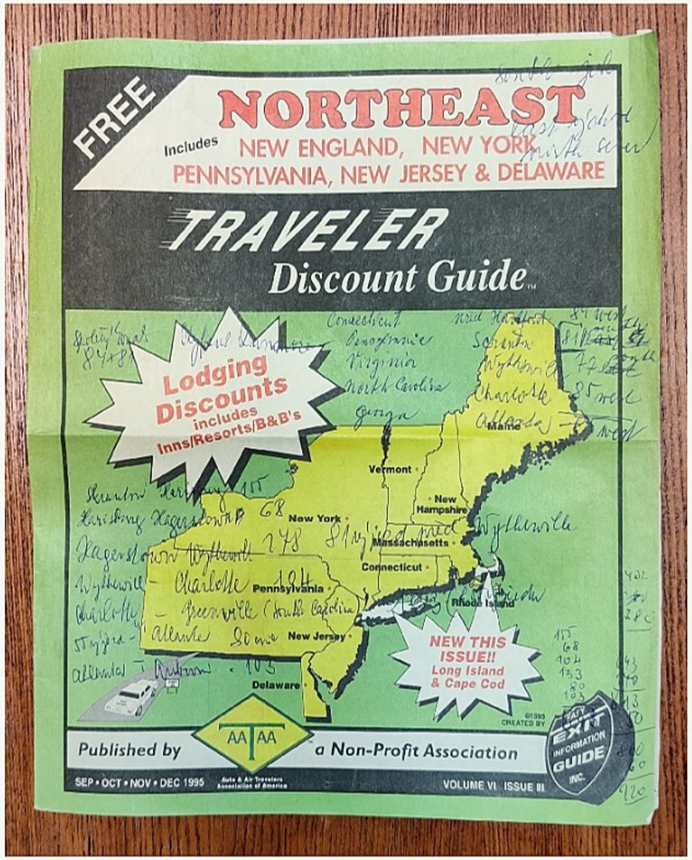
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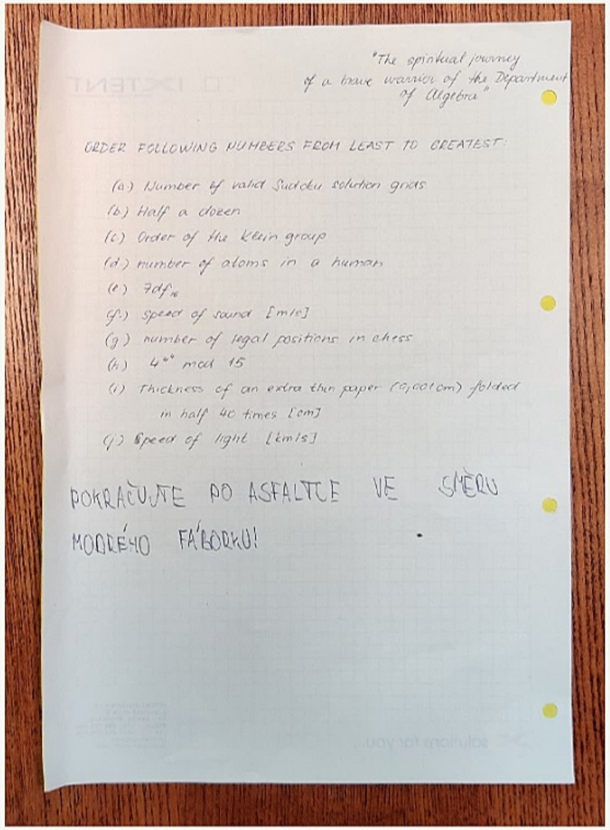
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(e) 7df16
(f) Speed of sound [m/s]
(g) number of legal positions in chess
(h) 4^40 mod 15
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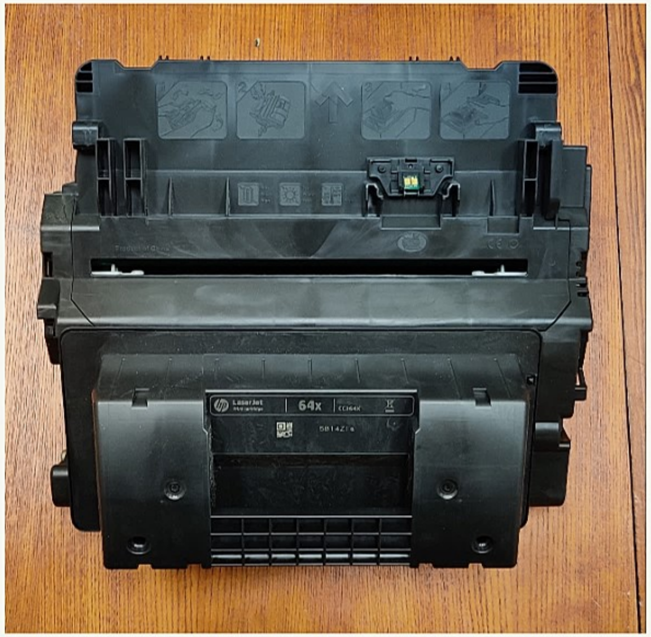

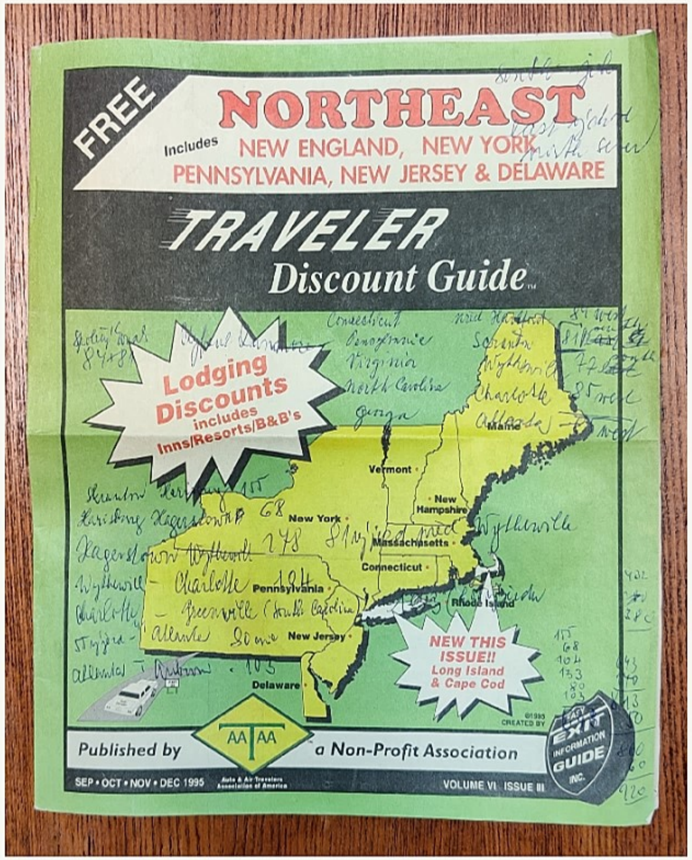
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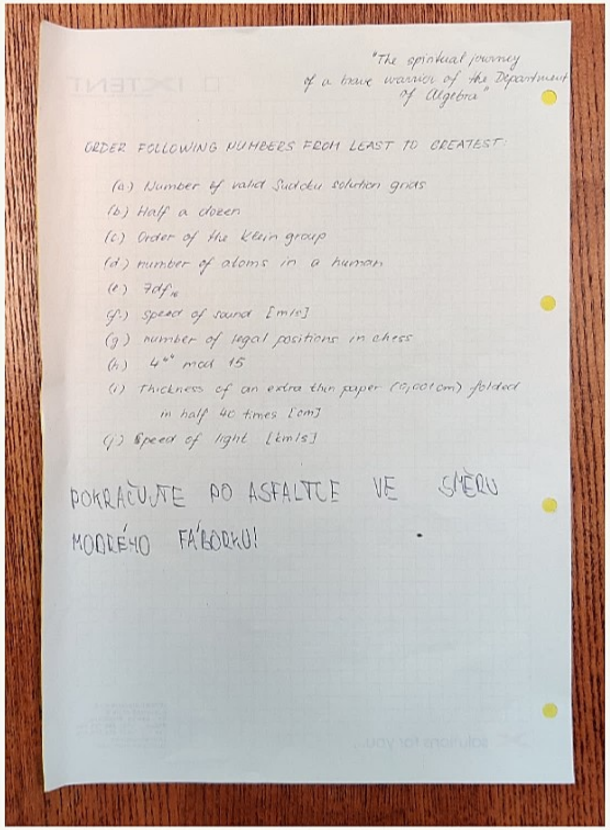
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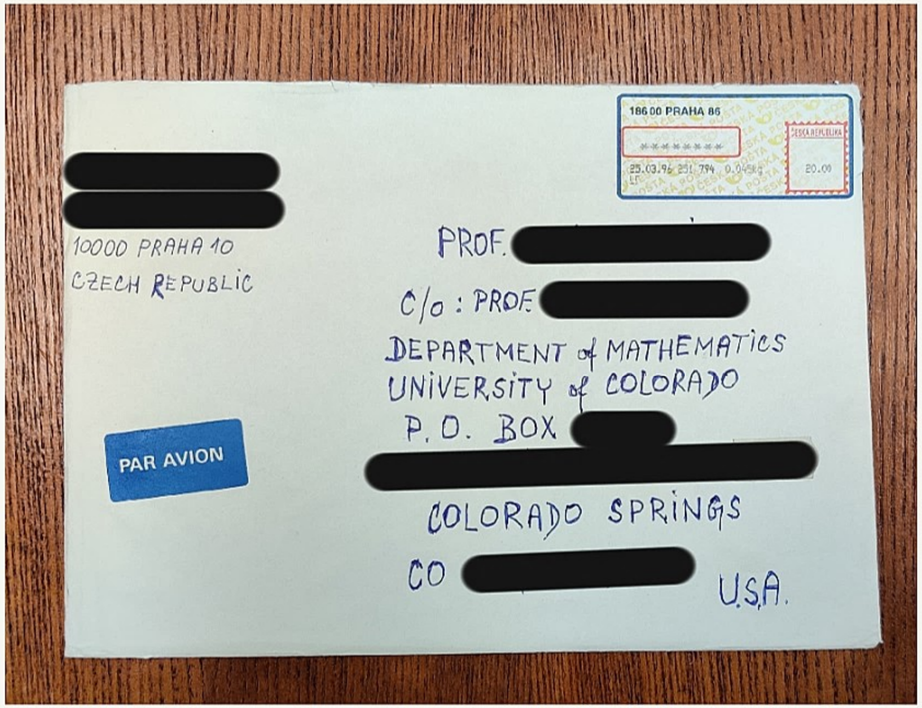
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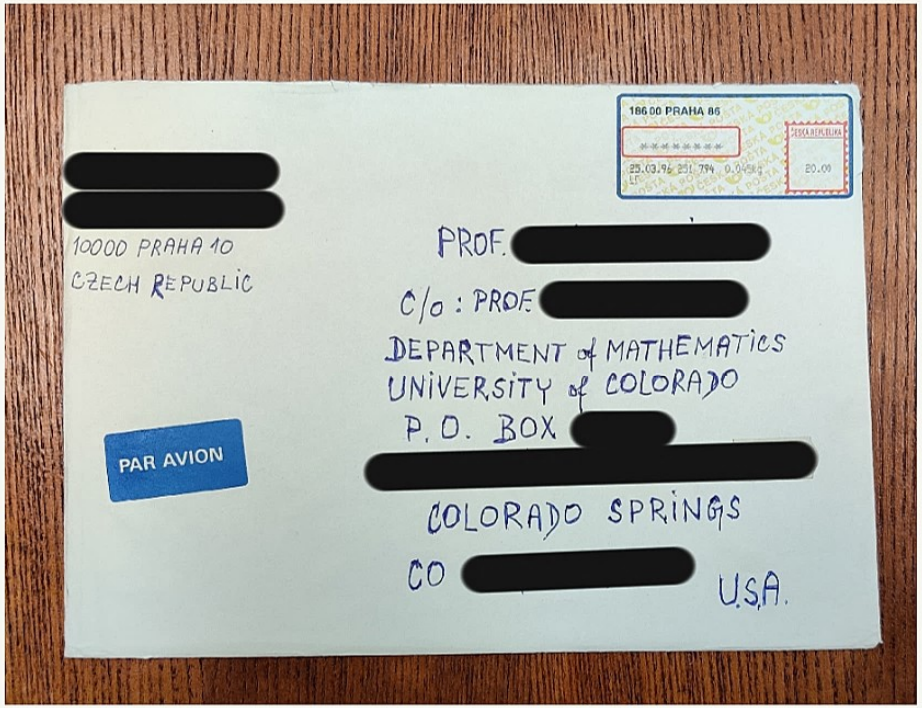
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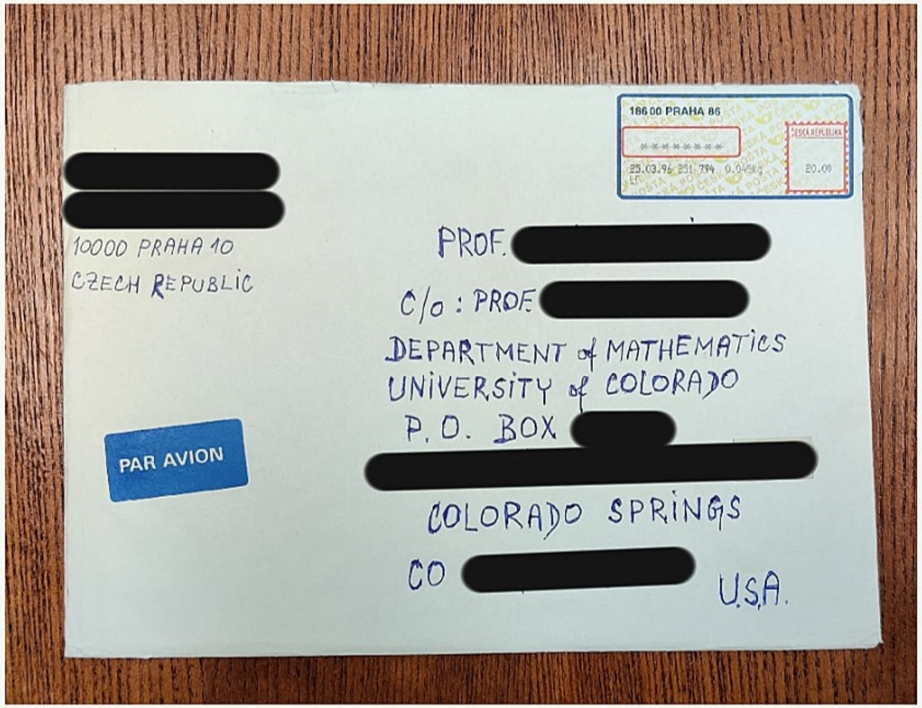
BIRTHDAY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HAPPY
BIRTHDAY


PF 96


PF 96

DISORDER = UNPREDICTABILITY

DISORDER IN ALGEBRA

	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

LATIN SQUARE

- max disorder
- min predictability

	1	2	3	4	5
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5

PROJECTION

- min disorder
- max predictability

QUANDLES

<u>DEF.</u> Q – non empty set, $\triangleright : Q^2 \to Q$ – binary operation.

$(Q, \triangleright)$ is a quandle if

(1) $\forall x \in Q : x \triangleright x = x$

(2) $\forall x, y \in Q \ \exists! z \in Q : z \triangleright x = y$

(3) $\forall x, y, z \in Q : (x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

QUANDLES

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EX. $(Q, x \triangleright y = x)$ is a projection quandle.

$(\mathbb{Z}_n, x \triangleright y = 2y - x)$ is a dihedral quandle.

QUANDLES AND GROUPS

PROP. $(Q, \triangleright)$ - quandle

(1) $(Q, \triangleright)$ has a neutral element $\rightarrow$ $|Q| = 1$.

(2) $(Q, \triangleright)$ is associative $\rightarrow$ $(Q, \triangleright)$ is a projection quandle.

QUANDLES AND GROUPS

PROP. $(Q, \triangleright)$ - quandle

(1) $(Q,\triangleright)$ has a neutral element $\rightarrow$ $|Q| = 1$.

(2) $(Q,\triangleright)$ is associative $\rightarrow$ $(Q,\triangleright)$ is a projection quandle.

RMK. Quandles are very far from being groups!

Compute : $3+6+4+6+4+6+4+6+4+6+4$.

WORKING WITHOUT ASSOCIATIVITY

Compute : $3+6+4+6+4+6+4+6+4+6+4$.

<u>With associativity</u> : $3+(6+4)+(6+4)+(6+4)+(6+4)+(6+4) = 53$.

<u>Without associativity</u> : $(((\cdots(3+6)+4)+6)+4)+6)+4)+6)+4)+6)+4) =$ 

WORKING WITHOUT ASSOCIATIVITY

Compute : $3+6+4+6+4+6+4+6+4+6+4$.

With associativity : $3+(6+4)+(6+4)+(6+4)+(6+4)+(6+4) = 53$.

Without associativity : $(((\cdots(3+6)+4)+6)+4)+6)+4)+6)+4)+6)+4) =$ 

MORAL OF THE STORY : Things still work, but everything is much more clunky.

ENTROPY - DPD

(discrete probability distribution)

DEF. $p = (p_1, \dots, p_n)$ - DPD $\left(p_i \in [0,1] \ \forall i, \ \sum_{i=1}^{n} p_i = 1 \right)$.

The (Shannon) entropy of p is

$$h(p) = -\sum_{i=1}^{n} p_i \log(p_i).$$

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$$h(p) = -\sum_{i=1}^{n} p_i \log(p_i).$$

PROP. (1) $h(p) \geq 0$

(2) $0 \leq h(p) \leq \log(n)$

(3) $h(p) = 0 \iff p_i = 1$ and $p_j = 0 \ \forall j \neq i$ (min disorder – max predictability)

(4) $h(p) = \log(n) \iff p_i = \frac{1}{n} \ \forall i$ (max disorder – min predictability)

ENTROPY – FUNCTIONS

DEF. $a : \{1,\dots,n\} \to \{1,\dots,n\}$ – function.

(It is a DPD)

(1) The distribution of a is $\hat{a} = \left(\frac{|a^{-1}(1)|}{n}, \dots, \frac{|a^{-1}(n)|}{n} \right)$.

(2) The entropy of a is $h(a) := h(\hat{a})$.

ENTROPY - FUNCTIONS

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(2) The entropy of a is $h(a) := h(\hat{a})$.

PROP. (1) $h(a) = 0 \iff a$ is constant

(2) $h(a) = \log(n) \iff a$ is a permutation

ENTROPY — QUANDLES

DEF. $(Q,\triangleright)$ – quandle, $L_1,\dots,L_n$ – rows of $(Q,\triangleright)$.

The entropy of $(Q,\triangleright)$ is

$$H(Q) := \frac{1}{n}\sum_{i=1}^{n} h(L_i)$$

ENTROPY — QUANDLES

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RMK. (1) Why did we use the rows and not the columns?

(2) What does $H(Q)$ actually measure?

ENTROPY – QUANDLES

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<u>RMK</u>. (1) Why did we use the rows and not the columns?

(2) What does $H(Q)$ actually measure?

<u>PROP</u>. (1) $H(Q) = 0 \iff Q$ is a projection quandle

(2) $H(Q) = \log(n) \iff Q$ is a latin quandle ← (A quandle whose table is a latin square)

EXAMPLES OF ENTROPY

EX. $Q = (\mathbb{Z}_n, x \triangleright y = 2y - x)$ – dihedral quandle.

$$H(Q) = \begin{cases} \log(n) & n \text{ odd} \\ \frac{1}{2}\log(n) & n \text{ even.} \end{cases}$$

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DEF. G – group. $\mathrm{Core}(G) := (G, x \triangleright y = yx^{-1}y)$ is a core quandle.

EXAMPLES OF ENTROPY

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DEF. G – group. $\mathrm{Core}(G) := (G, x \triangleright y = yx^{-1}y)$ is a core quandle.

$$H(\mathrm{Core}(D_n)) = \begin{cases} \log(2n) - \frac{n+1}{2n}\log(n+1) & n \text{ odd} \\ \log(2n) - \frac{1}{2n}\left[\left(\frac{n}{2}-1\right)\cdot 2\log 2 + (n+2)\log(n+2)\right] & n \text{ even.} \end{cases}$$

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Moreover, in any case

$$\lim_{n \to +\infty} H(\mathrm{Core}(D_n)) = +\infty$$

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(3) What quandle properties regulate its entropy?

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Ex.

	1	2	3	4	5	6
1	1	1	5	1	3	1
2	2	2	2	2	6	4
3	5	3	3	3	1	3
4	4	6	4	4	4	2
5	3	5	1	5	5	5
6	6	4	6	2	6	6

Core (D_3)

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Core (D_3)

Four equal elements and two different in every row

$H(Q) = h([1, 1, 5, 1, 3, 1])$

DEF. $(Q, \triangleright)$ – quandle. We define

(1) Right multiplication maps: $R_x : Q \to Q$, $q \mapsto q \triangleright x$ (columns)

(2) Left multiplication maps: $L_x : Q \to Q$, $q \mapsto x \triangleright q$ (rows)

Moreover

$$\mathrm{Inn}(Q) = \langle R_x : x \in Q \rangle \leq \mathrm{Sym}(Q)$$

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PROP. (1) $\mathrm{Inn}(Q) \trianglelefteq \mathrm{Aut}(Q) \leq \mathrm{Sym}(Q)$

(2) $\mathrm{Inn}(Q)$ and $\mathrm{Aut}(Q)$ act naturally on Q.

DEF. $(Q, \triangleright)$ - quandle

(1) If the action of $\mathrm{Inn}(Q)$ on Q is transitive, then Q is connected.

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Being homogeneous "pushes towards being ordered" (the entropy is constant on the rows).

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Being faithful "pushes towards being disordered". Does it exist a lower bound for the entropy of faithful quandles?

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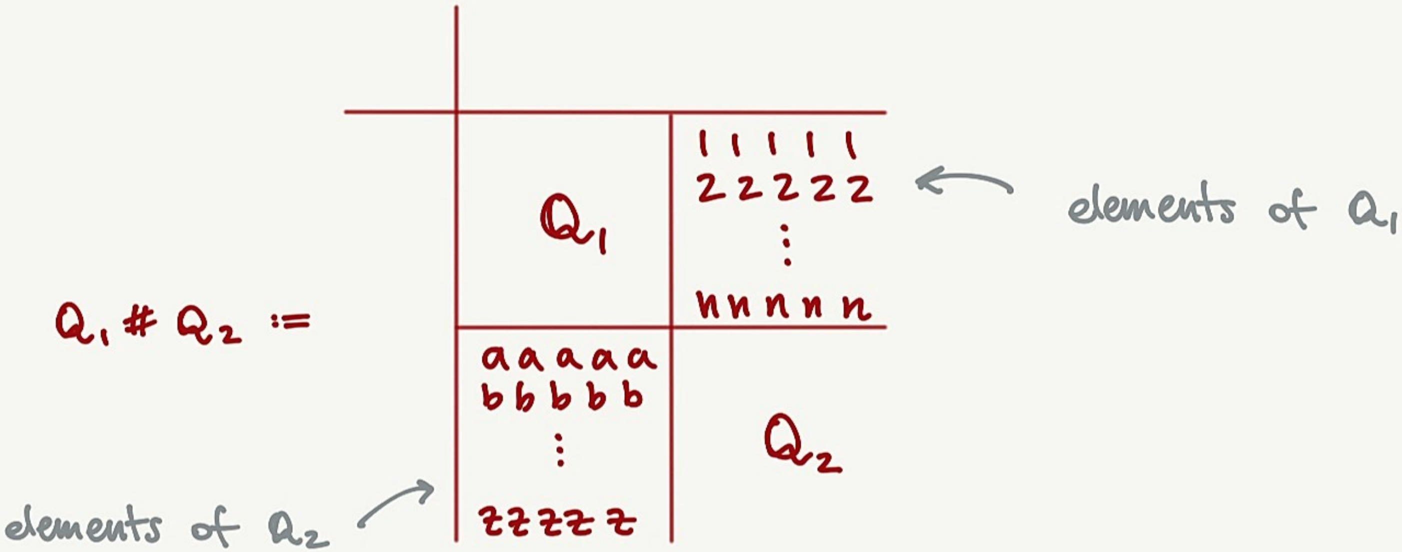
$Q_1 \# Q_2 :=$



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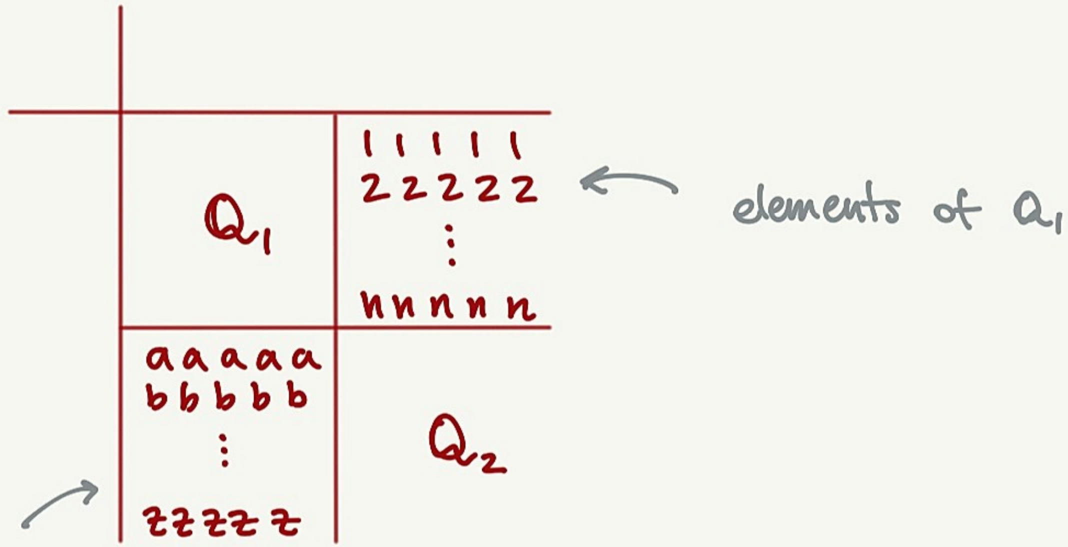

elements of Q_2

and the k^{th} multiple of a quandle Q : $kQ := \underbrace{Q \# Q \# \dots \# Q}_{k \text{ times}}$

PROP. (1) $\#$ is commutative and associative

(2) $Q_1 \# Q_2$ is a not connected quandle

(3) Q_1, Q_2 faithful $\Rightarrow$ $Q_1 \# Q_2$ faithful

(4) $H(kQ) \leq \log\left(1+\frac{1}{k-1}\right) + \frac{1}{k}\left[H(Q) + \log(k|Q|)\right].$

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RMK. There is no lower bound for the entropy of faithful quandles.

ENTROPY AND H, S, P OPERATORS

SURVEY

P $H(Q_1 \times Q_2) = \ldots$

(a) $H(Q_1) + H(Q_2)$

(b) $H(Q_1) \cdot H(Q_2)$

(c) $\max\{H(Q_1), H(Q_2)\}$

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PROP. $Q_1 \twoheadrightarrow Q_2 \rightarrow H(Q_2) \leq H(Q_1)$

COR. $H(Q/\theta) \leq H(Q)$.

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Unfortunately, none of the above.

Q_1	1	3	5
1	1	5	3
3	5	3	1
5	3	1	5

$\hookrightarrow$

Core (D_3)

Q_2	1	2	3	4	5	6
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2	2	2	2	2	6	4
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$H(Q_2) = \log 3$

$H(Q_2) = \log 3 - \frac{1}{2} \log 2$

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PROP. Q_1, Q_2 quandles, $|Q_1| = n$, $|Q_2| = m$, $Q_1 \hookrightarrow Q_2 \Rightarrow H(Q_1) \leq \frac{n^2}{m^2} \cdot H(Q_2)$

$\boxed{\#}$ $H(Q_1 \# Q_2) = \dots$

SURVEY

(a) $H(Q_1) + H(Q_2)$

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PROP. Q_1, Q_2 quandles, $|Q_1| = n$, $|Q_2| = m$.

$$H(Q_1 \# Q_2) \leq 1 + \left(\frac{n^2}{m^2+n^2} \right) H(Q_1) + \left(\frac{m^2}{m^2+n^2} \right) H(Q_2)$$

CONCLUSIONS

(*) The entropy function measures some properties of quandles and it behaves relatively well with respect to Universal Algebraic constructions.

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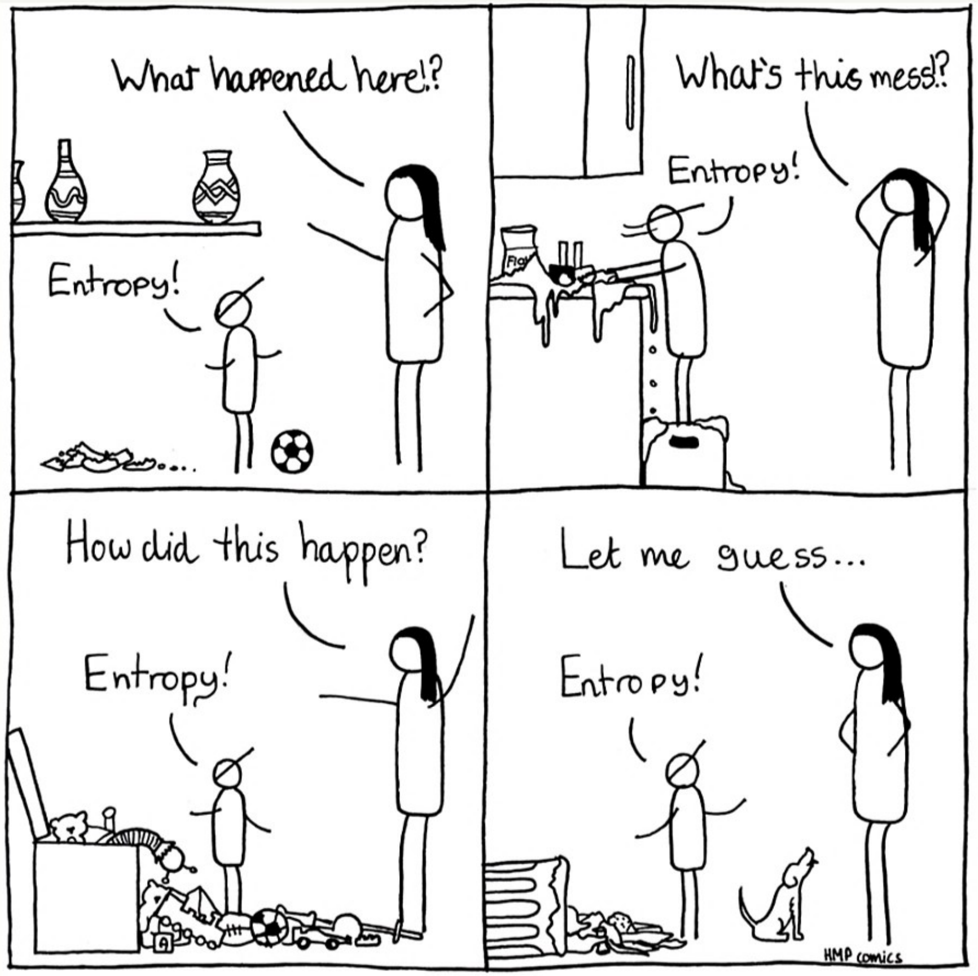
NEW HORIZONS:

(?) lower bound for connected, faithful, non-latin quandles.

THANKS FOR YOUR ATTENTION!

spaggiari@karlin.mff.cuni.cz



This is why we don't teach our children about entropy until much later...