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Mutually normalizing regular subgroups of the holomorph of C_{p^n}

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Let us introduce some fundamental notions.

Definition

Let G be a group. The **holomorph** of G is

$$\mathsf{Hol}(G) = \langle \mathsf{Aut}(G), \rho(G) \rangle \leq \mathsf{Sym}(G)$$

where $\rho(G) = \{\sigma_g \colon x \mapsto xg \mid g \in G\}$ is the subgroup of right multiplication maps.

Thus, the holomorph of a group is a very large subset of bijective maps.

What are these normalizing graphs?

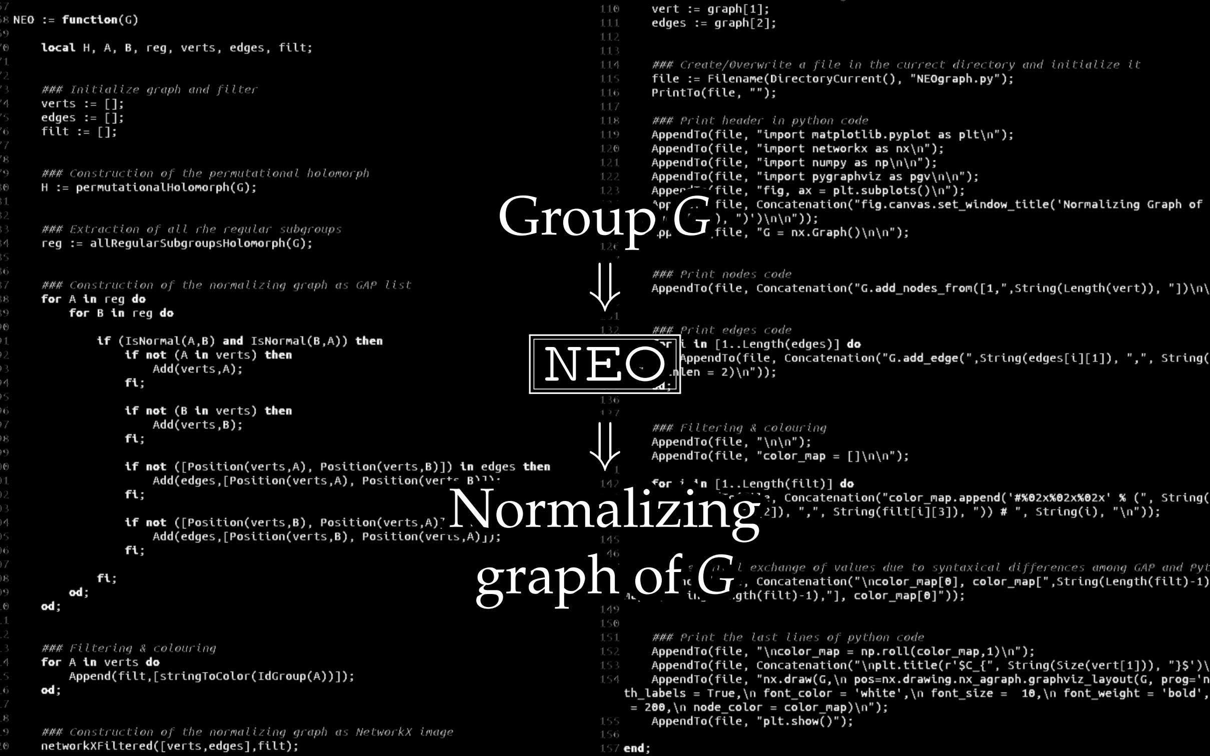
Definition

The **normalizing graph** of a group G is a graph where

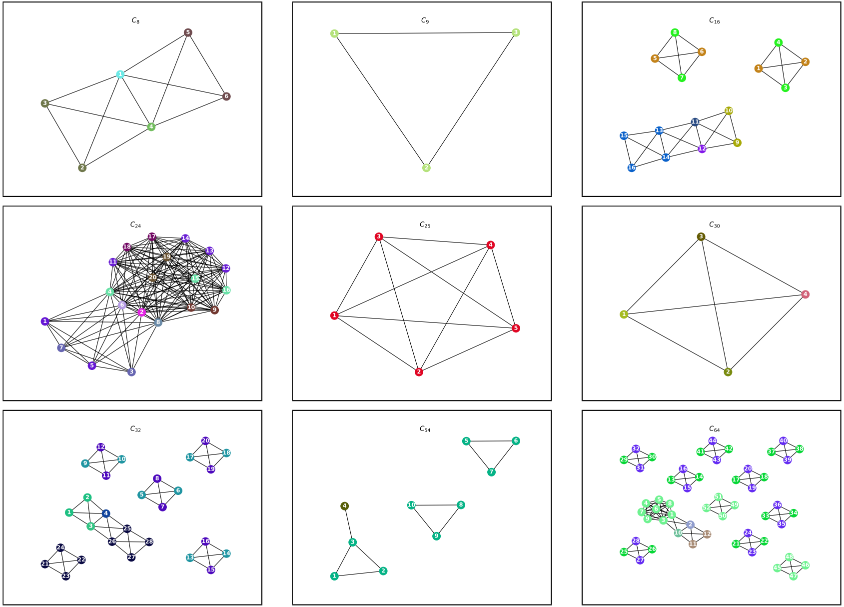
1. The *vertices* are the regular subgroups of $\mathsf{Hol}(G)$.
2. An *edge* represents a mutual normalization in $\mathsf{Sym}(G)$.

Motivation: It has several connections with the recent theory of *skew braces* and the *Yang-Baxter equation*.

The *pièce de résistance* of the coding part of this work is certainly the GAP function `NEO`.


Group G
⇓
NEO
⇓
Normalizing graph of G
NEO := function(G)
local H, A, B, reg, verts, edges, filt;
Initialize graph and filter
verts := [];
edges := [];
filt := [];
Construction of the permutational holomorph
H := permutationalHolomorph(G);
Extraction of all rhe regular subgroups
reg := allRegularSubgroupsHolomorph(G);
Construction of the normalizing graph as GAP list
for A in reg do
for B in reg do
if (IsNormal(A,B) and IsNormal(B,A)) then
if not (A in verts) then
Add(verts,A);
fi;
if not (B in verts) then
Add(verts,B);
fi;
if not ([Position(verts,A), Position(verts,B)]) in edges then
fi;
fi;
fi;
od;
od;
Filtering & colouring
for A in verts do
Append(filt,[stringToColor(IdGroup(A))]);
od;
Construction of the normalizing graph as NetworkX image
networkXFiltered([verts,edges],filt);
vert := graph[1];
edges := graph[2];
Create/Overwrite a file in the currect directory and initialize it
file := Filename(DirectoryCurrent(), "NEOgraph.py");
PrintTo(file, "");
Print header in python code
AppendTo(file, "import matplotlib.pyplot as plt\n");
AppendTo(file, "import networkx as nx\n");
AppendTo(file, "import numpy as np\n\n");
AppendTo(file, "import pygraphviz as pgv\n\n");
Print nodes code
Filtering & colouring
AppendTo(file, "\n\n");
AppendTo(file, "color_map = []\n\n");
Print the last lines of python code
AppendTo(file, "\ncolor_map = np.roll(color_map,1)\n");
AppendTo(file, "plt.show()");
end;

Can you spot the pattern?



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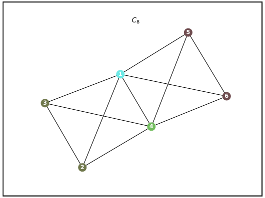

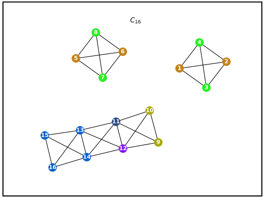

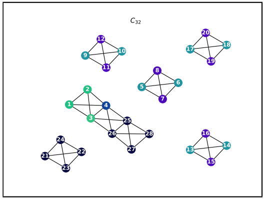

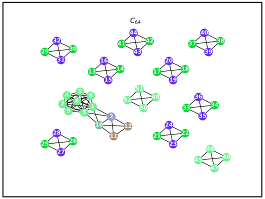

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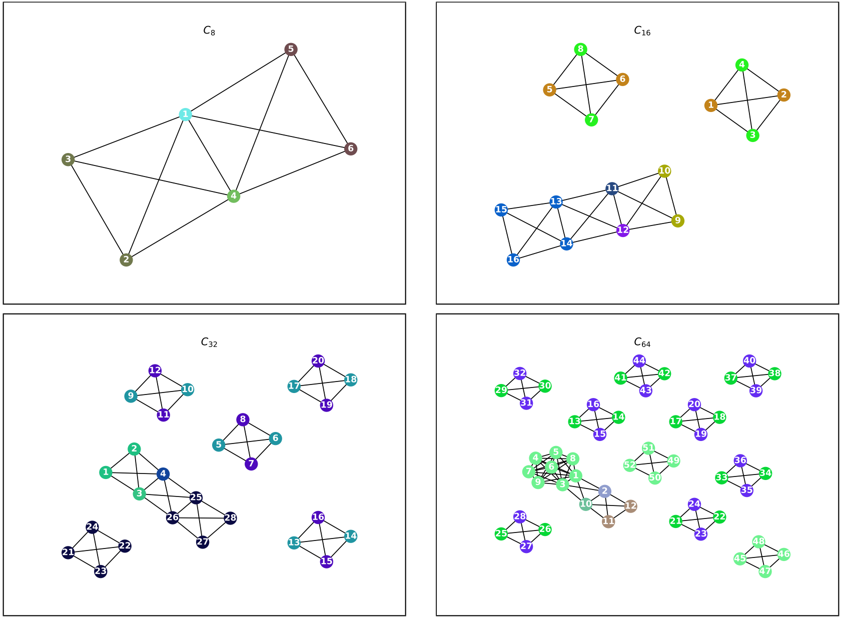


Problem

Find and prove the normalizing graph of C_{pn}

Different notions, same concept

Notation. For $x \in G$ and $\varphi \in \mathsf{Sym}(G)$ we denote by $x^\varphi = \varphi(x)$.

Theorem (A. Caranti, 2020 [1])

Let $(G, \cdot)$ be a finite group. The following data are equivalent.

1. *A regular subgroup $N \leq \mathsf{Hol}(G, \cdot)$.*
2. *A **gamma function** $\gamma \colon (G, \cdot) \to \mathsf{Aut}(G, \cdot)$, i.e. such that*
$$\gamma(x^{\gamma(y)} \cdot y) = \gamma(x)\gamma(y) \qquad \forall x, y \in G.$$
3. *A group operation $\circ$ on G such that $x \circ y = x^{\gamma(y)}$ for every $x, y \in G$.*

Expected question(s). How is N connected with γ and $\circ$?
Why are we introducing gamma functions?

The case $p = 2$

"two is the oddest prime number"

Guess some gamma functions...

After having used GAP to obtain some raw information...

In C_{16} we have

x	0; 8	1; 9	2; 10	3; 11	4; 12	5; 13	6; 14	7; 15
$\gamma(x)$	σ_1	σ_3	σ_5	σ_7	σ_9	σ_{11}	σ_{13}	σ_{15}

Guess:

$$\begin{aligned} \gamma \colon G &\to \mathsf{Aut}(G) \\ x &\mapsto \sigma_{2x+1} \end{aligned}$$

In the same way, we obtain the following gamma functions

Gamma function	Isomorphism class
$\gamma_1(x) = \sigma_1$	C_{2^n}
$\gamma_2(x) = \sigma^x_{2^{n-1}+1}$	C_{2^n}
$\gamma_3(x) = \sigma^x_{2^{n-1}-1}$	Q_{2^n}
$\gamma_4(x) = \sigma^x_{2^n-1}$	D_{2^n}
$\gamma_{\mathsf{p}}(x) = \sigma_{2x+1}$	$\mathsf{C}_2 \times \mathsf{C}_{2^{n-1}}$
$\gamma_{\mathsf{c},u}(x) = \sigma_{2^u x+1} \qquad u = 2, \dots, n$	C_{2^n}

Gamma function	Isomorphism class
$\gamma_5(x) = \begin{cases} \sigma_1 & x \equiv 0 \pmod 4 \\ \sigma_{2^{n-1}-1} & x \equiv 1 \pmod 4 \\ \sigma_{2^{n-1}+1} & x \equiv 2 \pmod 4 \\ \sigma_{2^n-1} & x \equiv 3 \pmod 4 \end{cases}$	SD_{2^n}
$\gamma_6(x) = \begin{cases} \sigma_1 & x \equiv 0 \pmod 4 \\ \sigma_{2^n-1} & x \equiv 1 \pmod 4 \\ \sigma_{2^{n-1}+1} & x \equiv 2 \pmod 4 \\ \sigma_{2^{n-1}-1} & x \equiv 3 \pmod 4 \end{cases}$	SD_{2^n}
$\gamma_{\mathsf{m}}(x) = \begin{cases} \sigma_{2x+1} & x \equiv 0 \pmod 2 \\ \sigma_{2x+2^{n-2}+1} & x \equiv 1 \pmod 2 \end{cases}$	M_{2^n}

Generate the others via conjugation

Roughly speaking, to **conjugate a gamma function** γ **by an automorphism** means simply to permute the elements of image of γ.

Notation. For a gamma function γ and $\sigma_{2k+1} \in \mathrm{Aut}(G)$ we denote by $\gamma^k = \gamma^{\sigma_{2k+1}^{-1}}$.

γ	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_{p}	γ_{m}	$\gamma_{\mathrm{c},u}$
$\lvert\gamma^{\mathrm{Aut}(G)}\rvert$	1	1	1	1	2	2	2^{n-2}	2^{n-2}	2^{n-u-1}

Proposition

There are **at least** $3 \cdot 2^{n-2} + 4$ regular subgroups in $\mathrm{Hol}(G)$.

Expected question. Why is this procedure called *conjugation*?

Uniqueness of the gamma functions

This was the most difficult part of the entire work. Proofs are long, technical and boring (at least, the proofs I found are so).

Mutual normalization problem

Idea: Translate the mutual normalization

Theorem

Let $(G, \cdot)$ be a group such that $\mathsf{Aut}(G)$ is abelian, and let $N, M \leq \mathsf{Hol}(G)$ be regular subgroups. Denote by

$$\gamma \colon (G, \circ) \to \mathsf{Aut}(G), \qquad \delta \colon (G, \bullet) \to \mathsf{Aut}(G)$$

respectively the gamma functions associated with N and M. Then N and M mutually normalize each other if and only if

$$\begin{cases} \gamma(x) = \gamma\left(x \cdot (y \circ x)^{-1} \cdot (x \bullet y)\right) \\ \delta(x) = \delta\left(x \cdot (y \bullet x)^{-1} \cdot (x \circ y)\right) \end{cases} \qquad \forall x, y \in G.$$

Remark. This is a general result. In particular, for cyclic groups, this is a pair of equation in modular arithmetic, since $\mathsf{C}_{2^n} \cong \mathbb{Z}/2^n\mathbb{Z}$.

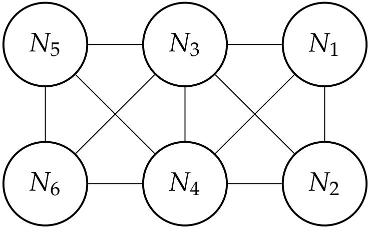
Mutual normalization of γ_i

Those conditions trivially hold for $\gamma_1, \ldots, \gamma_6$ in the following sense.

Corollary

$$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} \qquad \text{and} \qquad \{\gamma_3, \gamma_4, \gamma_5, \gamma_6\}$$

are mutually normalizing families of gamma functions.



Mutual normalization of γ_{c}

For a gamma function γ and $\sigma_{2k+1} \in \mathsf{Aut}(G)$ we denote by $\gamma^k = \gamma^{\sigma_{2k+1}^{-1}}$.

Proposition

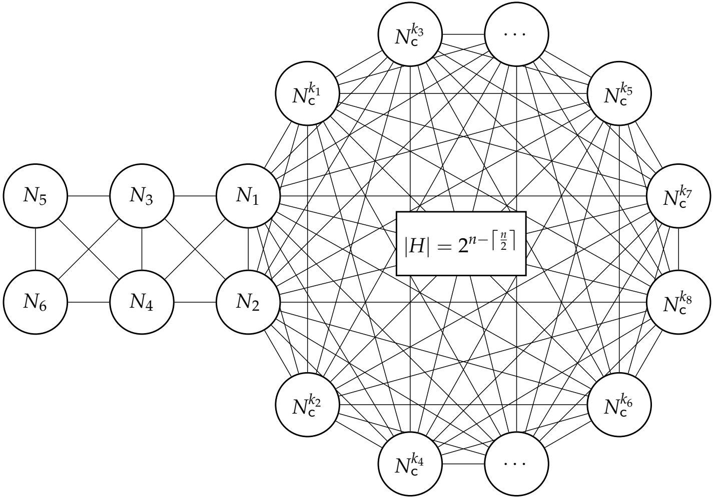
$$\gamma_{\mathsf{c},u}^{k} \rightleftharpoons \gamma_{\mathsf{c},v}^{h} \iff \begin{cases} 2^u(2k+1) \equiv 2^v(2h+1) \pmod{2^{n-u}} \\ 2^u(2k+1) \equiv 2^v(2h+1) \pmod{2^{n-v}} \end{cases}$$

Corollary

$$H = \left\{ \gamma_{\mathsf{c},u}^{k} \ : \ \left\lceil \frac{n}{2} \right\rceil \leq u \leq n \right\}$$

is composed by $2^{n-\lceil \frac{n}{2} \rceil}$ mutually normalizing gamma functions.

The comet


N_5
N_3
N_1
N_6
N_4
N_2
$N_{\mathsf{c}}^{k_1}$
$N_{\mathsf{c}}^{k_2}$
$N_{\mathsf{c}}^{k_3}$
$N_{\mathsf{c}}^{k_4}$
$N_{\mathsf{c}}^{k_5}$
$N_{\mathsf{c}}^{k_6}$
$N_{\mathsf{c}}^{k_7}$
$N_{\mathsf{c}}^{k_8}$
$\cdots$
$\cdots$
$|H| = 2^{n-\lceil \frac{n}{2} \rceil}$

Mutual normalization of γ_{c}

Corollary

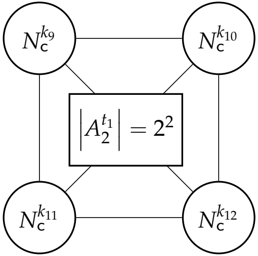
For every $2 \leq u < \lceil \frac{n}{2} \rceil$ and $0 \leq t < 2^{n-2u-1}$, the family

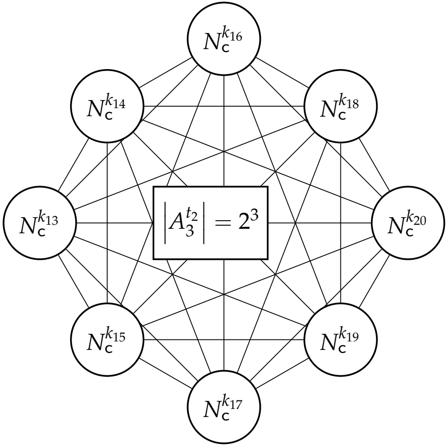
$$A_u^t = \left\{ \gamma_{\mathsf{c},u}^k \, : \, k \equiv t \pmod{2^{n-2u-1}} \right\}$$

is composed by 2^u mutually normalizing gamma functions. In total, there are

$$\frac{1}{3}\left(2^{n-3} - 2^{n-2\lceil \frac{n}{2} \rceil + 1}\right)$$

distinct A_u^t.


$N_c^{k_9}$
$N_c^{k_{10}}$
$\left|A_2^{t_1}\right| = 2^2$
$N_c^{k_{11}}$
$N_c^{k_{12}}$


$N_c^{k_{16}}$
$N_c^{k_{14}}$
$N_c^{k_{18}}$
$N_c^{k_{13}}$
$\left|A_3^{t_2}\right| = 2^3$
$N_c^{k_{20}}$
$N_c^{k_{15}}$
$N_c^{k_{19}}$
$N_c^{k_{17}}$

Mutual normalization of γ_{p} and γ_{m}

Proposition

$$\gamma_{\mathsf{p}}^k \rightleftharpoons \gamma_{\mathsf{p}}^h \iff k \equiv h \pmod{2^{n-3}}$$

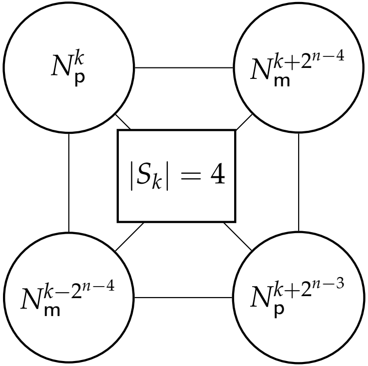
$$\gamma_{\mathsf{m}}^k \rightleftharpoons \gamma_{\mathsf{m}}^h \iff k \equiv h \pmod{2^{n-3}}$$

$$\gamma_{\mathsf{p}}^k \rightleftharpoons \gamma_{\mathsf{m}}^h \iff k - h \equiv 2^{n-4} \pmod{2^{n-3}}$$

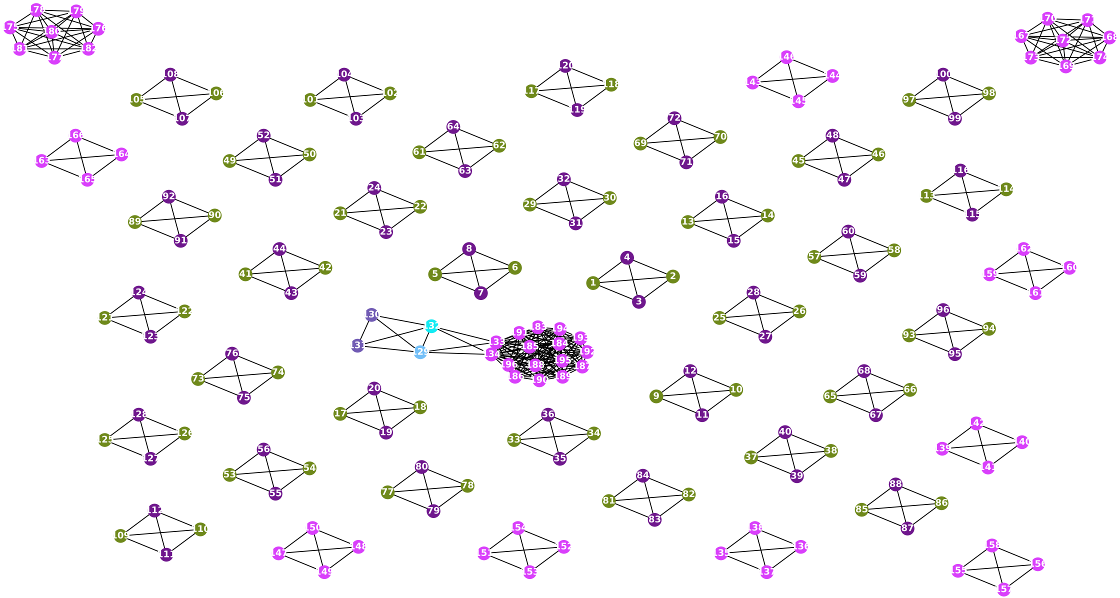
Corollary

$$S_k = \left\{ \gamma_{\mathsf{p}}^k, \gamma_{\mathsf{m}}^{k+2^{n-4}}, \gamma_{\mathsf{p}}^{k+2^{n-3}}, \gamma_{\mathsf{m}}^{k+2^{n-3}+2^{n-4}} \right\}$$

is composed by 4 mutually normalizing gamma functions. In total, there are 2^{n-3} distinct S_k.


N_{p}^{k}
$N_{\mathsf{m}}^{k+2^{n-4}}$
$|S_k| = 4$
$N_{\mathsf{m}}^{k-2^{n-4}}$
$N_{\mathsf{p}}^{k+2^{n-3}}$

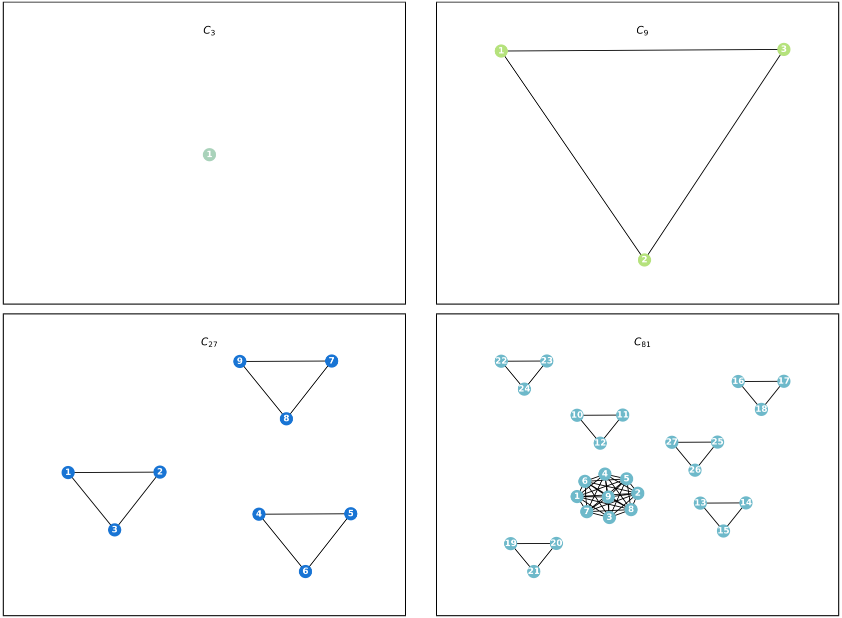
The local normalizing graph of C_{2n}



The case p odd

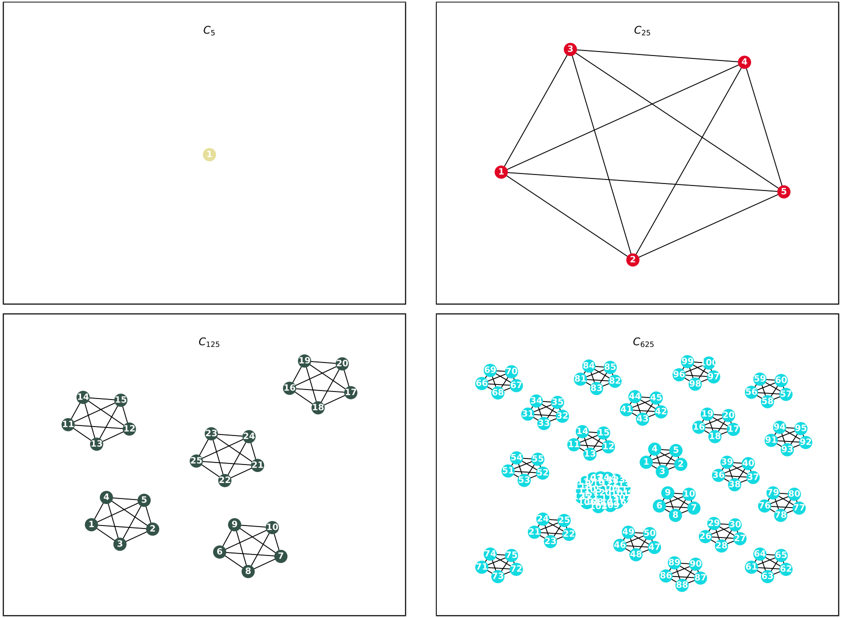
(A very quick look)

$p = 3$

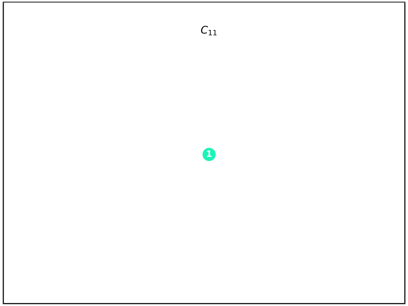


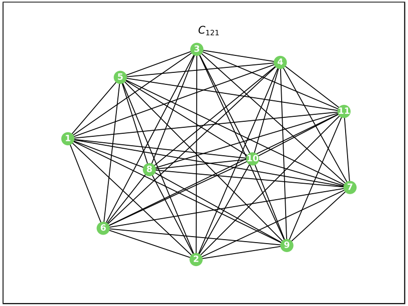
C_3
1
C_9
1
3
2
C_{27}
9
7
8
1
2
3
4
5
6
C_{81}
22
23
24
16
17
18
10
11
12
27
25
26
4
5
6
2
1
9
7
8
3
13
14
15
19
20
21

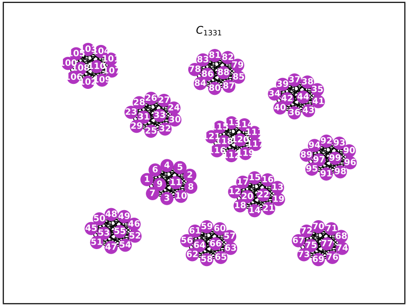
$p = 5$



$p = 11$





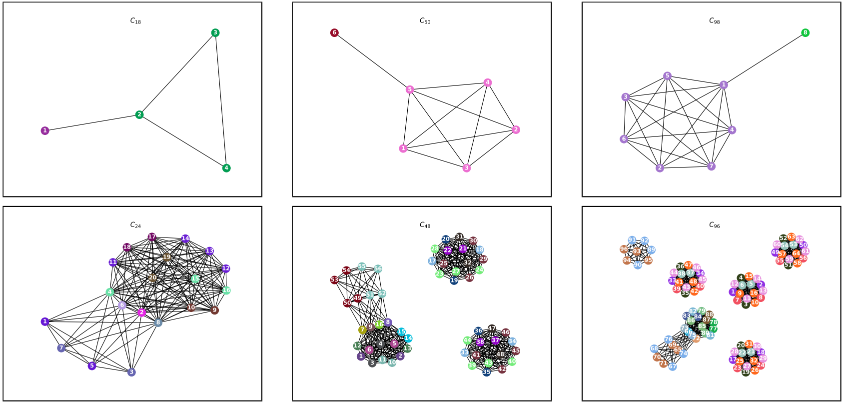


RAM MEMORY

NOT ENOUGH

Conclusion?

The mutually normalizing regular subgroups of $\mathrm{Hol}(C_{p^n})$ have been completely classified. Is it really time to be satisfied?



Ambition: We know that cyclic groups are the building blocks of *abelian groups*...

That’s all, thanks!

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