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Complex-quaternionic Analysis Applied to Spin-$\frac{1}{2}$ Massless Fields

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In the paper the complex-quaternionic analysis and its spinor version is used for the study of integral formulas for spin $\frac{1}{2}$ massless fields. The basic correspondence between the complexified Fueter equation and the massless field equation (spin $\frac{1}{2}$) is described first, together with the corresponding Cauchy integral formulas. It is shown then how the complexified Cauchy integral formula can be used to give the connection between elliptic type (boundary value type) integral formulas on Euclidean spacetime and Kirchhoff type (initial value type) integral formulas on Minkowski space (for spin $\frac{1}{2}$ massless fields). The explicit formulas showing such connection with the integral formula described by Penrose are given.

KEY WORDS complex-quaternionic analysis, massless fields, integral formulas

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1. INTRODUCTION

Quaternionic analysis ($\mathbb{H}$-analysis)—and more generally hypercomplex analysis—has made great progress in last decades (see, for example, [1]–[6]) and many properties of solutions of corresponding generalized Cauchy–Riemann equation are well understood now.

There are namely integral formulas for such solutions which are similar to the standard Cauchy integral formula in complex variable theory.

On the other hand, the explicit integral formula giving the solution to the (null cone) initial value problem was found by Penrose ([7]) and was the starting point for the description of massless fields within the twistor theory ([8]).

The main idea of the paper is to show that complex-quaternionic analysis (CH-analysis) can offer the integral formula which is a generalization of both mentioned, at the first sight quite different, integral formulas. Such approach was pioneered by Imaeda ([9]) who first pointed out the advantages of a complexification of quaternionic analysis (he is using the name biquaternions instead of complex quaternions) and who showed, how the quaternionic Cauchy integral formula can be complexified and used in the context of classical electrodynamics. I would like to stress, however, another point of view in this paper, namely that CH-analysis can be used, perhaps even more naturally, for the description of spin $\frac{1}{2}$ massless field on complex Minkowski space $\mathbb{C}M$. The complexified Cauchy integral formula then clearly have to have some meaning for spin $\frac{1}{2}$ massless fields on (real) Minkowski space, too. It is really the case, it will be shown that, after the integration with respect to a suitably chosen variable being done, the complexified Cauchy integral formula reduces to Penrose's integral formula.

The interest of such procedure lies in the fact that complexified Cauchy integral formula interpolates between two integral formulas quite different in character. The first one is the typical elliptic-type (or boundary-value type) integral formula for Fueter's regular function on $\mathbb{H}$:

$$F(P) = 1/2\pi^2 \int_{S_3} F(Q) \cdot DQ^{\dagger} \cdot (Q-P) \cdot 1/|Q-P|^4;$$

$$P \in \operatorname{Int} S_3 \subset \mathbb{H} \quad (1)$$

(for details see §2). The contour of integration is 3-dimensional and doesn't change with P; only values of F on S_3 are needed.

On the other hand, the integral formula of Penrose for spin $\frac{1}{2}$ massless field ϕ_A on Minkowski space is the typical hyperbolic-type (or initial value type) integral formula

$$\phi_A(P) = 1/2\pi \int_{\Sigma_2} (\zeta_A/r)\{D(\phi_A\xi^A) - (2\rho+\epsilon)(\phi_A\xi^A)\}\, dS \quad (2)$$

(for details see §5). The contour of integration Σ_2 is 2-dimensional here and depends on the point P (it is just the intersection of the initial value 3-surface with the null cone of the point P) and both values of the field and the derivative in a suitable direction are needed in the formula.

The procedure which converts the formula (1) into formula (2) is based on the deformation of the contour S_3 through complex Minkowski space (without crossing the singularities of the integrand in (1)) into the contour of the type $\Sigma_2 \times S_1$, where Σ_2 is from (2) and S_1's are small circles around Σ_2 (going into $\mathbb{C}M$ and avoiding so the singularity of (1)). This is the generalisation of the procedure given in [9]. The integration over S_1 can be carried over then using the standard residue theorem, the singularity of the integrand being just at the points of Σ_2. The singularity can be shown to be the pole of the order 2, hence the value of the field together with the first derivatives are needed. If the contour $\Sigma_2 \times S_1$ is chosen properly, the formula (1) reduces just to (2). The other choices of $\Sigma_2 \times S_1$ can give another integral formulas in Minkowski space, the Penrose's one, however, being the simplest one.

This paper is devoted only to spin $\frac{1}{2}$ case, but it is the reasonable conjecture that similar correspondence between (extended) $\mathbb{CH}$-analysis and massless fields on complex Minkowski space can be worked out for arbitrary spin. This possibility will be discussed in another paper.

The complexified Cauchy integral formula, together with the facts needed from $\mathbb{CH}$-analysis, is described in §2. The corresponding spinor version which is basically the translation of $\mathbb{CH}$-version by the representation of $\mathbb{CH}$ by 2×2 complex matrices, is presented in §3. It is preferred here, for the convenience of physicists, to keep §3 self-contained (i.e. without using results from quaternionic analysis), the effort needed to do so being very small. The general procedure of deformation of the contour of integration S_3 to the contour of the type $\Sigma_2 \times S_1$ is described in §4. In §5 the reduction of the elliptic-type integral formula to a hyperbolic-type one is computed in the case of Penrose's integral formula.

2. CAUCHY INTEGRAL FORMULA IN $\mathbb{CH}$

The $\mathbb{CH}$-analysis is simply the complexification of the standard quaternionic analysis, described in [2]. One reason why quaternionic analysis has to be complexified is the fact that the solutions of the basic Fueter's equation are real-analytic maps from $\mathbb{H}$ to $\mathbb{H}$. Hence a lot can be gained, as usually, if the complexified situation, where more complete information can be expected, is studied first and the restriction to the real case is considered then. Another, perhaps more important reason, is the fact that the direct connection to the problems of mathematical physics on Minkowski space is thus possible. We shall see, for example, that complexified Fueter equation is just spin $\frac{1}{2}$ massless field equation (considered on complex Minkowski space—see [8]). Only the facts necessary for sufficiently general Cauchy integral formula for $\mathbb{CH}$-functions will be described here, even if many other properties of regular quaternionic functions can be generalised to $\mathbb{CH}$ (see [9]) or even to complex Clifford algebra valued functions ([10]). Some pieces of new informations are added to the basically known facts (see [9], [10]).

Basic Notation

The complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ can be combined together to give the algebra $\mathbb{CH}$ of complex quaternions. The algebra $\mathbb{CH}$ is defined by $\mathbb{CH} := \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$. The typical member of $\mathbb{CH}$ can be written (after a choice of quaternionic units i_1, i_2, i_3 with $i_1 i_2 = i_3$; etc.) as $Q = Q_0 + i_1 Q_1 + i_2 Q_2 + i_3 Q_3$; $Q_0, Q_1, Q_2, Q_3 \in \mathbb{C}$. The quaternionic conjugation $\dagger$ and the norm (squared) will be given by $|Q|^2 := Q \cdot Q^\dagger$; $Q^\dagger := Q_0 - i_1 Q_1 - i_2 Q_2 - i_3 Q_3$. Both $\mathbb{C}$ and $\mathbb{H}$ will be supposed to be embedded into $\mathbb{CH}$ and we shall denote by $i\mathbb{H}$ the set $\{Q \in \mathbb{CH} \mid Q = iq,\ q \in \mathbb{H}\}$. The symbol $\mathbb{C}N_P$ will be used for the complex null cone $\{Q + P \mid |Q|^2 = 0\}$.

Differential forms on $\mathbb{CH}$ with coefficients in $\mathbb{CH}$ have to be introduced shortly. The $\mathbb{CH}$-valued r-form on $\mathbb{CH}$ will be defined to be an alternating $\mathbb{C}$-multilinear mapping from $\mathbb{CH} \times \cdots \times \mathbb{CH}$ to $\mathbb{CH}$. The exterior product of two such forms is defined by the standard formula ([2]; p. 203). A $\mathbb{CH}$-valued r-form ω can be always written as $\omega = \omega_0 + i_1\omega_1 + i_2\omega_2 + i_3\omega_3$; where $\omega_0, \ldots, \omega_3$ are the standard $\mathbb{C}$-valued r-forms. The exterior derivative d is defined to act on the components ω_i separately. Let us define a few interesting forms:

$$dQ := dQ_0 + i_1 dQ_1 + i_2 dQ_2 + i_3 dQ_3 .$$

$$dQ^{\dagger} := dQ_0 - i_1 dQ_1 - i_2 dQ_2 - i_3 dQ_3 .$$

$$\begin{aligned} DQ &:= dQ_1 \wedge dQ_2 \wedge dQ_3 - i_1 dQ_0 \wedge dQ_2 \wedge dQ_3 - i_2 dQ_0 \wedge dQ_3 \wedge dQ_1 \\ &\quad - i_3 dQ_0 \wedge dQ_1 \wedge dQ_2 \\ &= 1/3!\, dQ \wedge dQ^{\dagger} \wedge dQ \end{aligned}$$

$$\begin{aligned} DQ^{\dagger} &:= dQ_1 \wedge dQ_2 \wedge dQ_3 + i_1 dQ_0 \wedge dQ_2 \wedge dQ_3 + i_2 dQ_0 \wedge dQ_3 \wedge dQ_1 \\ &\quad + i_3 dQ_0 \wedge dQ_1 \wedge dQ_2 \\ &= 1/3!\, dQ^{\dagger} \wedge dQ \wedge dQ^{\dagger} . \end{aligned}$$

As in [2] we have

LEMMA 1 *For* $P, Q, R \in \mathbb{CH}$ *we have*: (i) $DQ[P, Q, R] = \frac{1}{2}(R \cdot P^{\dagger} \cdot Q - Q \cdot P^{\dagger} \cdot R)$ (ii) $DQ^{\dagger}[P, Q, R] = \frac{1}{2}(Q^{\dagger} \cdot P \cdot R^{\dagger} - R^{\dagger} \cdot P \cdot Q^{\dagger}) = (DQ[P, Q, R])^{\dagger}$

Proof Both left and right hand sides are—as maps from $\mathbb{CH} \times \mathbb{CH} \times \mathbb{CH}$ to $\mathbb{CH}$—the complexifications of the corresponding mappings $\mathbb{H} \times \mathbb{H} \times \mathbb{H}$ to $\mathbb{H}$ from [2]. Hence the part (i) follows from [2], Prop. 1, p. 204. The part (ii) can be proved in the same way.

Differential forms will be integrated over chains in $\mathbb{CH}$. The cubical differential singular chains will be used here (i.e: differentiable maps of the unit interval $I^r \subset \mathbb{R}^r$ into $\mathbb{CH}$—see [11], p. 252). The integration of a $\mathbb{CH}$-valued differential form $\omega = \omega_0 + i_1\omega_1 + i_2\omega_2 + i_3\omega_3$ is done componentwise. Generalized Stokes theorem ([11], p. 256) can be clearly used for $\mathbb{CH}$-valued forms.

There are two basic differential operators and 4 different forms of the complexified Fueter equation. Let us define

$$\partial := \partial/\partial Q_0 + i_1\partial/\partial Q_1 + i_2\partial/\partial Q_2 + i_3\partial/\partial Q_3 ;$$

$$\partial^{\dagger} := \partial/\partial Q_0 - i_1\partial/\partial Q_1 - i_2\partial/\partial Q_2 - i_3\partial/\partial Q_3 .$$

Differential equations (acting on the mapping $F(Q) = F_0(Q) + i_1F_1(Q) + i_2F_2(Q) + i_3F_3(Q)$) are:

(i) $\partial \cdot F^{+} = 0$ (it is the complexification of $\bar{\partial}_L f = 0$ of [2])
(ii) $\partial^{+} \cdot F = 0$
(iii) $F^{+} \cdot \partial = 0$ (the derivatives are acting to the left)
(iv) $F \cdot \partial^{+} = 0$

Note We shall always suppose in the paper that all mappings are holomorphic with respect to Q_0, Q_1, Q_2, Q_3.

Example The function $G(Q) = Q/|Q|^4$ satisfies $\partial^{+} \cdot G = G \cdot \partial^{+} = 0$ and $\partial \cdot G^{+} = G^{+} \cdot \partial = 0$ for $|Q|^2 \neq 0$.

LEMMA 2 *We have (in $\Omega \subset \mathbb{CH}$):*
(1) $\partial \cdot G^{+} = 0,\ F^{+} \cdot \partial = 0 \Rightarrow d(F^{+} \cdot DQ \cdot G^{+}) = 0$; (2) $\partial^{+} \cdot G = 0,\ F \cdot \partial^{+} = 0 \Rightarrow d(F \cdot DQ^{+} \cdot G) = 0$.

Proof In the case (1) we have

$$DQ \cdot G^{+} = G^{+} dQ_1 \wedge dQ_2 \wedge dQ_3 - i_1 G^{+} dQ_0 \wedge dQ_2 \wedge dQ_3 - i_2 G^{+} dQ_0 \wedge dQ_3 \wedge dQ_1 - i_3 G^{+} dQ_0 \wedge dQ_1 \wedge dQ_2,$$

hence

$$d(DQ \cdot G^{+}) = \{\partial G^{+}/\partial Q_0 + i_1 \partial G^{+}/\partial Q_1 + i_2 \partial G^{+}/\partial Q_2 + i_3 \partial G^{+}/\partial Q_3\} \times dQ_0 \wedge dQ_1 \wedge dQ_2 \wedge dQ_3 = (\partial \cdot G^{+}) dQ_0 \wedge dQ_1 \wedge dQ_2 \wedge dQ_3.$$

The same is true for $F^{+} \cdot DQ$, so we have

$$d(F^{+} \cdot DQ \cdot G^{+}) = \left[(F^{+}\partial)G^{+} + F^{+}(\partial G^{+})\right] dQ_0 \wedge dQ_1 \wedge dQ_2 \wedge dQ_3.$$

THEOREM 1 (Cauchy) *Let $\Omega \subset \mathbb{CH}$ be an open set, let Σ_3 and Σ_3' be compact chains, which belong to the same homology class in Ω (i.e. $\Sigma_3 - \Sigma_3' = \partial u$ for a 4-chain in Ω). Then:*

(i) $F^{+} \cdot \partial = 0,\ \partial \cdot G^{+} = 0 \Rightarrow \int_{\Sigma_3} F^{+} \cdot DQ \cdot G^{+} = \int_{\Sigma_3'} F^{+} \cdot DQ \cdot G^{+}$
(ii) $F \cdot \partial^{+} = 0,\ \partial^{+} \cdot G = 0 \Rightarrow \int_{\Sigma_3} F \cdot DQ^{+} \cdot G = \int_{\Sigma_3'} F \cdot DQ^{+} \cdot G.$

Remarks 1. Some useful information is contained in the case $F \equiv 1$ or $G \equiv 1$. 2. In the case of noncompact chains some integrability conditions on F, G should be added.

Proof The Stokes theorem gives

$$\int_{\Sigma_3 - \Sigma_3'} F^\dagger \cdot DQ \cdot G^\dagger = \int_u d(F^\dagger \cdot DQ \cdot G^\dagger) = 0.$$

Let us consider now a point $P \in \mathbb{CH}$ and 4-dimensional (positively oriented) ball $U_\rho(P) = \{P + Q \mid Q \in \mathbb{H}, |Q|^2 \leqslant \rho^2\}$, which lies in the (real) quaternionic space $\mathbb{H}$ shifted to the point P. Let us denote by S_3 the boundary $\partial U_\rho(P)$. Then we have

THEOREM 2 (Cauchy integral formula) *Let $\Omega \subset \mathbb{CH}$ be an open set, suppose that $P \in U_\rho(P) \subset \Omega$ and let Σ_3 be a cycle, homological in $\Omega \backslash \mathbb{C}N_P$ with the sphere S_3. Then*

(i) $F^\dagger \cdot \partial = 0$ *in* $\Omega \Rightarrow \lambda$
$F^\dagger(P) = 1/2\pi^2 \int_{\Sigma_3} F^\dagger(Q) \cdot DQ \cdot (Q - P)^\dagger / |Q - P|^4.$

(ii) $F \cdot \partial^\dagger = 0$ *in* $\Omega \Rightarrow \lambda$
$F(P) = 1/2\pi^2 \int_{\Sigma_3} F(Q) \cdot DQ^\dagger \cdot (Q - P)/|Q - P|^4.$

(iii) $\partial^\dagger \cdot G = 0$ *in* $\Omega \Rightarrow \lambda$
$G(P) = 1/2\pi^2 \int_{\Sigma_3} \{(Q - P)/|Q - P|^4\} \cdot DQ^\dagger \cdot G(Q).$

(iv) $\partial \cdot G^\dagger = 0$ *in* $\Omega \Rightarrow \lambda$
$G^\dagger(P) = 1/2\pi^2 \int_{\Sigma_3} \{(Q - P)^\dagger / |Q - P|^4\} \cdot DQ \cdot G^\dagger(Q).$

Proof We can always shift the point P to the origin by $Q' = Q - P$, $DQ' = DQ$ and $\tilde{F}(Q') = F(Q)$, hence we shall suppose $P = 0$. Then $S_3 \subset \mathbb{H}$ and Cauchy integral formula for regular quaternionic functions ([2], Th. 3) together with Theorem 1 gives the assertion.

COROLLARY *The Theorem 2 holds in the same form for $\tilde{S}_3 \subset i\mathbb{H}$, i.e. if $P \in i\mathbb{H}$, $\tilde{U}_\rho(P) = \{P + Q \mid Q \in i\mathbb{H}, -|Q|^2 \leqslant \rho^2\}$ and $\tilde{S}_3$ is the boundary of $\tilde{U}_\rho(P)$, then the assertion of the Theorem 2 holds in the same form for every cycle Σ_3, homological with $\tilde{S}_3$ in Ω.*

Proof It is sufficient to show that $F(P) = 1/2\pi^2 \int_{\tilde{S}_3} F(Q) \cdot D^\dagger Q \cdot (Q - P)/|Q - P|^4$. For it we can use Theorem 2 for $\tilde{F}(S) := F(iS)$, where $S = (-i)Q$, $R = (-i)P$ together with $i(S - R)/|S - R|^4 = (P - Q)/|P - Q|^4$ and $i^3 \cdot DS^\dagger = DQ^\dagger$.

THEOREM 3 *Let Σ_3 be a compact chain in $\mathbb{CH}$, let us denote $\Omega := \mathbb{CH}\setminus\{\bigcup_{P\in\Sigma_3} CN_P\}$. Suppose that f is a continuous function on Σ_3 with values in $\mathbb{CH}$. Then the function $F(P) = 1/2\pi^2\int_{\Sigma_3} f(Q)\cdot DQ^+ \cdot (Q-P)/|Q-P|^4$ satisfies the equation $F\cdot\partial^+ = 0$ in Ω.*

Remarks 1. The same is true, of course, for other 3 basic equations. 2. If Σ_3 is not compact, a suitable restriction on the behavior of f at ∞ has to be imposed.

Proof The standard theorem on the derivative of the integral depending on parameters can be applied here, hence the theorem follows from $(Q-P)/|Q-P|^4\cdot\partial^+ = 0$.

3. THE SPINOR VERSION OF THE $\mathbb{CH}$-CAUCHY INTEGRAL FORMULA

To get a connection with problems in mathematical physics, let us now translate the information contained in §2 into spinor language. For this it is sufficient simply to use the basic representation of $\mathbb{CH}$ by the algebra $\mathbb{C}(2)$ of 2×2 complex matrices. The standard rules of the spinor calculus will be used freely (see e.g. [12], [13]). The basic identification for the translation from $\mathbb{CH}$ to spinor language for complex Minkowski space $\mathbb{C}M$ will be the correspondence

$$z_\mu = [z_0, z_1, z_2, z_3] \in \mathbb{C}M \leftrightarrow Q = z_0 + ii_1z_1 + ii_2z_2 + ii_3z_3 \in \mathbb{CH}$$

$$\leftrightarrow z_{AA'} = \begin{bmatrix} z_0+z_3 & z_1 - iz_2 \\ z_1+iz_2 & z_0 - z_3 \end{bmatrix} \in \mathbb{C}(2)$$

Under this identification Q^+ corresponds to z^μ and to $z^{AA'}$; $|Q|^2$ to $z_\mu z^\mu$ and to $\det(z_{AA'}) = \frac{1}{2}(z_{AA'}z^{AA'})$. To translate the operators ∂, ∂^+, the quaternionic coefficients will be replaced by the corresponding 2×2 matrices and $\partial/\partial Q_i$ will be substituted by the corresponding derivatives $\nabla_{AA'} := \partial/\partial z^{AA'}$ (resp. $\partial_\mu := \partial/\partial z^\mu$). The formulas look like

$$\partial \leftrightarrow \nabla_{AA'} := \begin{bmatrix} \nabla_{00'} & \nabla_{10'} \\ \nabla_{01'} & \nabla_{11'} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \partial_0+\partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{bmatrix}, \qquad \partial^+ \leftrightarrow \nabla^{AA'}.$$

A function $F(Q) = F_0(Q) + i_1F_1(Q) + i_2F_2(Q) + i_3F_3(Q)$ will correspond to the mapping $\phi_{AA'} = \phi_{AA'}(z_{BB'})$ under the identification

$F_0 + i_1F_1 + i_2F_2 + i_3F_3 \leftrightarrow \phi_{AA'}$, $Q \leftrightarrow z_{BB'}$. The translation of four basic differential equations now looks like:

(i) $\nabla_{AA'}\phi^{AB'} = 0$;
(ii) $\nabla^{AA'}\phi_{BA'} = 0$;
(iii) $\nabla_{AA'}\phi^{BA'} = 0$;
(iv) $\nabla^{AA'}\phi_{AB'} = 0$.

All fields considered will be supposed to be holomorphic with respect to z_μ on their domain of definition.

Let us note that the spinor form of the equations can be separated into two independents parts. It is basically the consequence of the properties of matrix multiplication:

$$\partial \cdot F^\dagger = 0 \leftrightarrow \nabla_{AA'}\phi^{AB'} = 0 \leftrightarrow \begin{bmatrix} \nabla_{00'} & \nabla_{10'} \\ \nabla_{01'} & \nabla_{11'} \end{bmatrix} \cdot \begin{bmatrix} \phi^{00'} \\ \phi^{10'} \end{bmatrix} \begin{bmatrix} \phi^{01'} \\ \phi^{11'} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence the equation $\partial \cdot F^\dagger = 0$ consists from two independent copies of the spin $\frac{1}{2}$ massless field equation $\nabla_{AA'}\psi^{A'}$ on $\mathbb{C}M$. (Let us remark that such property is not very unpleasant because of the fact that the basic spinor fields in Nature often come in doublets—see [14].)

To translate differential forms on $\mathbb{CH}$, it is sufficient to interpret such an r-form as a mapping from $\mathbb{C}(2) \times \cdots \times \mathbb{C}(2) \rightarrow \mathbb{C}(2)$ instead of $\mathbb{CH} \times \cdots \times \mathbb{CH} \rightarrow \mathbb{CH}$, or—what is the same—as a 2×2 matrix the elements of which are standard r-forms on $\mathbb{C}M$.

With the abbreviations $d_\mu := dz_\mu$, $d_{AA'} := dz_{AA'}$ let us define

$$Dz_{AA'} := \frac{1}{2}\begin{bmatrix} d_{00'} \wedge d_{01'} \wedge d_{10'}\,;\ -d_{00'} \wedge d_{10'} \wedge d_{11'} \\ d_{00'} \wedge d_{01'} \wedge d_{11'}\,;\ -d_{01'} \wedge d_{10'} \wedge d_{11'} \end{bmatrix};$$

$$Dz^{AA'} := \frac{1}{2}\begin{bmatrix} -d_{01'} \wedge d_{10'} \wedge d_{11'}\,;\ d_{00'} \wedge d_{10'} \wedge d_{11'} \\ -d_{00'} \wedge d_{01'} \wedge d_{11'}\,;\ d_{00'} \wedge d_{01'} \wedge d_{10'} \end{bmatrix}.$$

Then we have

Lemma 1′

(i) $Dz_{AA'}[r_{BB'}, t_{CC'}, s_{DD'}] = \frac{1}{2}(s_{BA'}r^{BB'}t_{AB'} - t_{BA'}r^{BB'}s_{AB'})$.
(ii) $Dz^{AA'}[r_{BB'}, t_{CC'}, s_{DD'}] = \frac{1}{2}(t^{AB'}r_{BB'}s^{BA'} - s^{AB'}r_{BB'}t^{BA'})$.

Proof By direct computation, for example the upper left corner of the left hand side of (i) looks like

$$\tfrac{1}{2}\{r_{00'}t_{01'}s_{10'} + t_{00'}s_{01'}r_{10'} + s_{00'}r_{01'}t_{10'}$$

$$- s_{00'}t_{01'}r_{10'} - t_{00'}r_{01'}s_{10'} - r_{00'}s_{01'}t_{10'}\}$$

while the same corner on the right is

$$\tfrac{1}{2}\{s_{00'}r_{11'}t_{00'} - s_{10'}r_{01'}t_{00'} - s_{00'}r_{10'}t_{01'} + s_{10'}r_{00'}t_{01'}$$

$$- t_{00'}r_{11'}s_{00'} + t_{10'}r_{01'}s_{00'} + t_{00'}r_{10'}s_{01'} - t_{10'}r_{00'}s_{01'}\}.$$

Remark It follows from Lemma 1′ that $Dz_{AA'}$ is really a translation of DQ in the sense of alternating mappings.

LEMMA 2′ *We have (in* $\Omega \subset \mathbb{C}M$) $\nabla^{AA'}\phi_A = 0,\ \nabla^{AA'}\psi_{A'} = 0 \Rightarrow d(\phi_A\psi_{A'}Dz^{AA'}) = 0.$

Proof Using $df = \nabla^{AA'}f \cdot d_{AA'}$ we get for example $d(\phi_A Dz^{AA'}) = \tfrac{1}{2}\{-\nabla^{00'}\phi_0 - \nabla^{10'}\phi_1;\ -\nabla^{01'}\phi_0 - \nabla^{11'}\phi_1\}d_{00'} \wedge d_{01'} \wedge d_{10'} \wedge d_{11'}$. Hence

$$\nabla^{AA'}\phi_A = 0,\ \nabla^{AA'}\psi_{A'} = 0 \ in\ \Omega \Rightarrow \int_{\Sigma_3}\phi_A\psi_{A'}Dz^{AA'} = \int_{\Sigma'_3}\phi_A\psi_{A'}Dz^{AA'}.$$

THEOREM 1′ (Cauchy) *Let* $\Omega \subset \mathbb{C}M$ *be an open set, let* Σ_3 *and* Σ'_3 *be compact chains, which belong to the same homology class in* Ω. *Then*

$$\nabla^{AA'}\phi_A = 0,\ \nabla^{AA'}\psi_{A'} = 0 \ in\ \Omega \Rightarrow \int_{\Sigma_3}\phi_A\psi_{A'}Dz^{AA'} = \int_{\Sigma'_3}\phi_A\psi_{A'}Dz^{AA'}.$$

Proof It follows from Lemma 2′ and Stokes theorem.

Let us consider now the point $P_{AA'}$ in $\mathbb{C}M$ and the (positively oriented) ball $U_\rho(P) = \{P_\mu + Q_\mu \mid Q_\mu = [iy_0, x_1, x_2, x_3],\ y_0^2 + x_1^2 + x_2^2 + x_3^2 \leq \rho^2\}$ (i.e. $U_\rho(P)$ is the ball in Euclidean spacetime shifted to P). Let us denote by S_3 its boundary $\partial U_\rho(P)$. Then we have

THEOREM 2′ (Cauchy integral formula) *Let* $P \in U_\rho(P) \subset \Omega \subset \mathbb{C}M$, *let* Σ_3 *belongs to the same class of homology as* S_3 *(in* $\Omega \setminus \mathbb{C}N_P$*). Then*

$$\nabla^{AA'}\phi_A = 0 \text{ in } \Omega$$

$$\Rightarrow \phi_A(P) = 1/2\pi^2 \int_{\Sigma_3} \phi_B(Q) DQ^{BA'}(Q_{AA'} - P_{AA'})/|Q - P|^4.$$

where $|Q - P|^2 = \det(Q_{AA'} - P_{AA'})$.

Remark Both theorems have, of course, corresponding versions for $\phi^A, \psi^{A'}$.

Proof Let us give the proof which does not depend on quaternionic analysis. Theorem 1′ gives the independence of the integral on the representant of the homological class. The limit procedure $\rho \to 0$ can be applied and we get for this limit the value

$$\phi_B(P) \cdot 1/2\pi^2 \int_{S_3(\rho)} DQ^{BA'}(Q_{AA'} - P_{AA'})/|Q - P|^4 = \phi_B(P) \cdot k_A^B,$$

where the matrix k_A^B doesn't depend on ϕ_A. Hence the only difficult point in the proof is to compute k_A^B and to show that it is just the unit matrix. It is done in Example in §5.

THEOREM 3′ *Let* Σ_3 *be a compact chain in* $\mathbb{C}_M$*, let us denote* $\Omega = \mathbb{C}M \setminus \{\bigcup_{P \in \Sigma_3} \mathbb{C}N_P\}$*, where* $\mathbb{C}N_P$ *is the complex null cone of the point P. Let* θ_A *be continuous on* Σ_3*, then the field*

$$\phi_A(P) = 1/2\pi^2 \int_{\Sigma_3} \theta_B(Q) DQ^{BB'}(Q_{AB'} - P_{AB'})/|Q - P|^4 \quad (3)$$

satisfies the equation $\nabla^{AA'}\phi_A = 0$ *in* Ω.

Proof It is sufficient to use the standard theorems on derivatives of the integral depending on parameters and the fact that $\nabla^{AA'}\{(Q_{AB'} - P_{AB'})/|Q - P|^4\} = 0$.

Remark Theorem 3′ can be used to get some information on the analytic continuation of fields from Euclidean spacetime $\mathbb{R}_4$ to Minkowski spacetime. For example, if a field ϕ_A, satisfying massless field equation on a ball $U = \{[iy_0, x_1, x_2, x_3] \mid y_0^2 + x^2 < \rho^2\} \subset \mathbb{R}_4$ is given, then the field ϕ_A can be extended to the whole domain

$\in CM \mid C\mathcal{N}_P \cap \mathbb{R}_4 \subset \Omega\}$, by the formula (3). For a fixed P the cycle Σ_3 has to be a sphere in $\tilde{U}$, sufficiently close to the boundary of $\tilde{U}$. The rough estimate gives, for example, that the domain $\{z_\mu = x_\mu + iy_\mu \in CM \mid \sqrt{(y_0^2 + \mathbf{x}^2)} + \sqrt{(x_0^2 + \mathbf{y}^2)} < \rho\}$ is contained in $\tilde{U}$. From this it follows, for example, that massless fields ϕ_A, given on the whole $\mathbb{R}_4$, can be extended to the whole CM.

4. THE DEFORMATION OF THE CONTOUR OF INTEGRATION IN CM

Consider the complex null cone $C\mathcal{N}$ (with vertex at 0). Let us denote

$$\Sigma_3 := \{z_{AA'} = -r\zeta_A\bar{\zeta}_{A'}/(|\zeta_0|^2 + |\zeta_1|^2) + zv_{AA'} \mid$$

$$\zeta_A \in \mathbb{C}_2, z \in \mathbb{C}, |z| = \rho\}. \quad (4)$$

where $r = r(\zeta_A) > 0$ and $v_{AA'} = v_{AA'}(\zeta_B)$ are (smooth) functions on $P_1(\mathbb{C}) \simeq S_2$ described by homogeneous coordinates ξ_A. We want to show that under suitable assumptions on $v_{AA'}$ the cycle Σ_3 belongs to the same homology class (in $CM \setminus C\mathcal{N}$) as the sphere S_3 in $\mathbb{R}_4$. To prove this, it is sufficient to show that there is a homotopy in $CM \setminus C\mathcal{N}$ joining S_3 and Σ_3. The explicit description of such homotopy will be given now.

The main idea is shown in Fig. 1 (two space dimensions are suppressed):

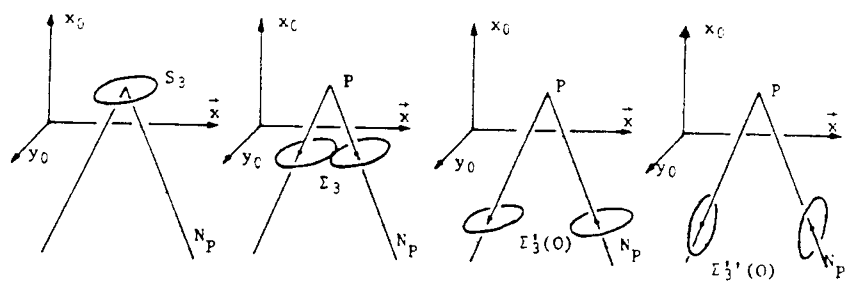

Figure 1.

The small circles, obtained by splitting of the original circle, are rotated to the z_0-plane (and possibly further to any other complex 1-dimensional subspace of $\mathbb{C}M$, which can be different for different points of Σ_3).

The full-dimensional case is rotationally symmetric version of the Fig. 1. The circle in S_3, corresponding to a direction $\pm\mathbf{n}$, $\mathbf{n} \in S_2$ will be splitting into two small circles around the two opposite generators of the null cone N (corresponding to the directions $\pm\mathbf{n}$). The best possible illustration is the transformation of the sphere into an apple and then to a torus.

The explicit formulas (in several steps) for such deformation look like (the condition on $v_{AA'} \simeq v_\mu$ will be discussed in a while):

Step 1 (from sphere to torus): Let $r > \rho > 0$. Define for $t \in \langle 0, r\rangle$

$$\Sigma_3(t) = \{ z_\mu \in \mathbb{C}M \mid z_0 = i\rho \sin\phi - t,\ \mathbf{x} = \mathbf{n}(\rho\cos\phi - t);$$
$$\mathbf{n} \in S_2,\ \phi \in \langle -\alpha, \alpha \rangle \},$$

where α satisfies $t + \rho\cos\alpha = 0$ for $t \leqslant \rho$ and $\alpha = \pi$ for $t > \rho$. Clearly $\Sigma_3(0) = S_3$.

Step 2 (the circles are turning to z_0-plane): $t \in \langle 0, 1\rangle$ we define

$$\Sigma_3'(t) = \{ z_\mu \in \mathbb{C}M \mid z_0 = i\rho\sin\phi - r + t\rho\cos\phi,$$
$$\mathbf{x} = -\mathbf{n}r + (1-t)\rho\mathbf{n}\cos\phi;\ \mathbf{n} \in S_2,\ \phi \in \langle -\pi, \pi \rangle \}.$$

Then $\Sigma_3'(0) = \Sigma_3(r)$ and $\Sigma_3'(1)$ can be written as

$$\Sigma_3'(1) = \{ z_\mu \mid z_0 = -r + z,\ \mathbf{x} = -\mathbf{n}r;\ \mathbf{n} \in S_2,\ z \in \mathbb{C},\ |z| = \rho \}.$$

Step 3 (independent rotating of circles): Let us define for $t \in \langle 0, 1\rangle$

$$\Sigma_3''(t) = \{ z_\mu \in \mathbb{C}M \mid z_0 = -r + z[(1-t) + tv_0],$$
$$\mathbf{x} = -\mathbf{n}r + zt\mathbf{v};\ \mathbf{n} \in S_2,\ z \in \mathbb{C},\ |z| = \rho \}.$$

Then $\Sigma_3''(0) = \Sigma_3'(1)$ and $\Sigma_3''(1)$ looks in the spinor notation like

$$\Sigma_3''(1) = \left\{ -r\left[1/(|\zeta_0|^2 + |\zeta_1|^2) \right] \zeta_A \bar{\zeta}_{A'} + z v_{AA'} ;\ \zeta_A \in \mathbb{C}_2,\ z \in \mathbb{C},\ |z| = \rho \right\}.$$

The centers of the circles can be now shifted along generators of the null cone arbitrarily, so r can be (by homotopy $r(1-t) + r(\zeta)t$, $t \in \langle 0, 1 \rangle$) changed into any function $r(\zeta)$.

The last point missing yet is the verification that the described homotopy is really the homotopy in $\mathbb{C}M \setminus \mathbb{C}\mathcal{N}$. Some conditions on v_μ have to be imposed to ensure this. Let us note first that all deformations are enclosed in the space $\{z_\mu \mid z_0 \in \mathbb{C};\ z_1, z_2, z_3 \in \mathbb{R}\}$. But it is easy to see that the complex null cone intersects this space in Minkowski space null cone $\mathcal{N}$. Hence only points in (real) Minkowski space have to be checked. In step 1 and 2 it means $\sin\varphi = 0$ and the verification is easy. For the step 3 more care is needed. The question is if the points $q_\mu \pm z[(1+t)\tau_\mu + tv_\mu]$, $t \in \langle 0, 1 \rangle$, $|z| = \rho$ small lie on $\mathcal{N}$, where $\tau_\mu = [1:0:0:0]$, $q_\mu = [-r, -\mathbf{n}r]$ (see Fig. 2).

The condition which is sufficient to exclude such possibility is the assumption that τ_μ and v_μ lie in the same component of $M \setminus T_q\mathcal{N}$; i.e. that $q_\mu \tau^\mu$ and $q_\mu v^\mu$ have the same sign. But $q_\mu \tau^\mu = q_0 < 0$, hence we shall suppose (in the spinor language) that $\zeta_A \bar{\zeta}_{A'} v^{AA'} > 0$.

Lemma 3 *Suppose that the function $v^{AA'}(\zeta)$ satisfies the condition $\zeta_A \bar{\zeta}_{A'} v^{AA'}(\zeta) > 0$, $\zeta_A \in \mathbb{C}_2$. Let U be a neighborhood of the set $\{z_{AA'} = -t\zeta_A\bar{\zeta}_{A'} \mid \zeta_A \in \mathbb{C}_2,\ t \in \langle 0, r(\zeta) \rangle\}$ in $\mathbb{C}M$.*

Then for $\rho > 0$ sufficiently small the contour Σ_3 belongs in $U \setminus \mathbb{C}\mathcal{N}$ to the same homology class as S_3.

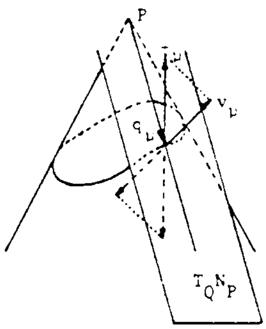

Figure 2.

Proof The condition $\zeta_A \bar{\zeta}_{A'} v^{AA'} > 0$ holds uniformly on the considered compact subset of N, hence, choosing ρ sufficiently small, the step 3 can be checked. The radius ρ can be chosen so small that the whole deformation is contained in U.

5. THE INTEGRAL FORMULA OF PENROSE

A generalized Kirchhoff integral formula giving the values of massless fields by means of the initial value data on a null hypersurface $\mathcal{N}$ in $\mathbb{C}M$ was described in [7], [15]. It can be described as follows:

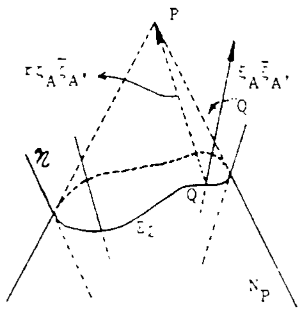

Figure 3.

Let Σ_2 be the intersection of the (backward) null cone N_P of the point P with the initial data null hypersurface $\mathcal{N} \subset M$. Take a typical point Q' on Σ_2. Suppose that $\zeta_A \in \mathbb{C}_2$ are homogeneous coordinates for $\mathbb{P}_1(\mathbb{C})$ worth of null directions in N_P and suppose that the vector $Q'P$ is described in the spinor language as $r\zeta_A\bar{\zeta}_{A'}$, $r = r(\zeta_A) > 0$. Let us choose further a (smooth) spinor field $\xi_A = \xi_A(Q')$ on the hypersurface $\mathcal{N}$ such that the vector $\xi_A\bar{\xi}_{A'}$ is tangent to the generator of $\mathcal{N}$ (i.e. that is orthogonal to $T_{Q'}\mathcal{N}$). We can suppose the normalisation condition $\xi^A\zeta_A = 1$ on Σ_2. Then

$$\phi_A(P) = 1/2\pi \int_{\Sigma_2} (\zeta_A/r)\{D\theta - (2\rho + \epsilon)\theta\}\, dS, \tag{5}$$

where $\theta = \phi_A\xi^A$, $D \equiv \xi^A\bar{\xi}^{A'}\nabla_{AA'}$, $\rho = -\xi^A\zeta^B\xi^{B'}\nabla_{BB'}\xi_A$, $\epsilon = -\zeta^A D\xi_A$ and dS is the surface element of Σ_2.

We want to show now that the complexified Cauchy integral formula reduces to Penrose's formula on M after one integration being done. Suppose that the field ϕ_A satisfies the equation $\nabla^{AA'}\phi_A = 0$ in a neighborhood $U \subset CM$ of the set $\{z_{AA'} = P_{AA'} - t\zeta_A\bar{\zeta}_{A'} \mid \zeta_A \in C_2,\ t \in \langle 0, r\rangle\} \subset N$. The contour $\Sigma_3 = \{Q_{AA'} = P_{AA'} - r\zeta_A\bar{\zeta}_{A'} + z\xi_A\bar{\xi}_{A'} \mid \zeta_A \in C_2,\ z \in C,\ |z| = \rho\}$ satisfies (for ρ sufficiently small) the conditions of Theorem 2' so that we have $\phi_A(P) = 1/2\pi^2\int_{\Sigma_3}\phi_B(Q)\, DQ^{BA'}(Q_{AA'} - P_{AA'})/|Q - P|^4$.

Let us transform now the 3-form under the integral into variables $\zeta \in C,\ z \in C,\ |z| = \rho$, where ζ is the corresponding inhomogeneous coordinate, i.e. $\zeta_A = [1, \zeta]$. Then

$$DQ^{BA'} = DQ^{BA'}\left[\partial Q_{CC'}/\partial\zeta, \partial Q_{DD'}/\partial z, \partial Q_{EE'}/\partial\bar{\zeta}\right] d\zeta \wedge dz \wedge d\bar{\zeta}.$$

$$Q_{AA'} - P_{AA'} = -r\zeta_A\bar{\zeta}_{A'} + z\xi_A\bar{\xi}_{A'}, \qquad |Q - P|^2 = -zr.$$

Hence

$$\phi_A(P) = 1/2\pi^2\int\int\oint\phi_B(Q)$$

$$\times\frac{1}{2}\Big\{\xi^B\bar{\xi}^{C'}\Big[\big(-\partial/\partial\zeta(r\zeta_C)\bar{\zeta}_{C'} + z\partial/\partial\zeta(\xi_C\bar{\xi}_{C'})\Big]$$

$$\times\Big[\big(-\partial/\partial\bar{\zeta}(r\bar{\zeta}^{A'})\zeta^C + z\partial/\partial\bar{\zeta}(\bar{\xi}^{A'}\xi^C)\big)\Big]$$

$$\Big[\big(-\partial/\partial\bar{\zeta}(r\bar{\zeta}^{C'})\zeta^B + z\partial/\partial\bar{\zeta}(\bar{\xi}^{C'}\xi^B)\big)\Big]$$

$$\times\Big[\big(-\partial/\partial\zeta(r\zeta_C)\bar{\zeta}_{C'} + z\partial/\partial\zeta(\xi_C\bar{\xi}_{C'}\big)\Big]\bar{\xi}^{A'}\xi^C\Big\}$$

$$\cdot\left(-r\zeta_A\bar{\zeta}_{A'} + z\xi_A\bar{\xi}_{A'}\right)/z^2r^2 \cdot dz \wedge d\bar{\zeta} \wedge d\zeta.$$

The integrand looks very complicated and there is even a lot of derivatives and multiplications to be done yet (it represents the sum of 64 different terms). But the miraculous cancellation will take place and very simple formula will be obtained at the end. The basic tool for cancellations is, of course, $\zeta_A\zeta^A = \xi_A\xi^A = 0$, but it is not sufficient, the important information is contained in the formula $\partial/\partial\zeta(r\zeta_B)\xi^B = 0$ (and its primed version), which expresses the fact that the vector $\xi_A\bar{\xi}_{A'}$ is orthogonal to $T_Q\Sigma_2$. Moreover, only terms containing $1/z$ will contribute by residue theorem. Note that $\phi_B(P_{DD'} - r\zeta_D\bar{\zeta}_{D'} + z\xi_D\bar{\xi}_{D'})$ has to be expanded to Taylor series at $z = 0$. Altogether we find

$$\mathrm{res}_0 = -\tfrac{1}{2}(D\phi_B)\xi^B\bar{\xi}^{C'}\bar{\zeta}_{C'}\left[\partial/\partial\zeta(r\zeta_C)\zeta^C\right]\left[\partial/\partial\bar{\zeta}(r\bar{\xi}^{A'})\bar{\zeta}_{A'}\right]\zeta_A/zr$$

$$+\tfrac{1}{2}\phi_B\xi^B\bar{\xi}^{C'}\bar{\zeta}_{C'}\left[\partial/\partial\zeta(r\zeta_C)\partial/\partial\bar{\zeta}(\xi^C)\right]\bar{\xi}^{A'}\bar{\zeta}_{A'}\zeta_A/zr$$

$$+\tfrac{1}{2}\phi_B\xi^B\bar{\xi}^{C'}\partial/\partial\zeta(\bar{\xi}_{C'})\xi_C\zeta^C\left(\partial/\partial\bar{\zeta}(r\bar{\xi}^{A'})\bar{\zeta}_{A'}\zeta_A/zr\right.$$

$$= \tfrac{1}{2}\Big\{(D\phi_B)\xi^B r^2 + \theta\left[\partial/\partial\zeta(r\zeta_C)\partial/\partial\bar{\zeta}(\xi^C)\right]$$

$$-\theta r\left(\bar{\xi}^{C'}\partial/\partial\xi\zeta(\bar{\xi}_{C'})\right)\Big\}\zeta_A/zr.$$

Further, the matrix $\xi^A\zeta_B - \zeta^A\xi_B$ is just the unit matrix (because of $\zeta_A\xi^A = 1$), hence we have $\partial/\partial\zeta(r\zeta_C) = -(\partial/\partial\zeta(r\zeta_D)\zeta^D)\xi_C = r\xi_C$ and writing $\phi_C = \phi_C(z_{EE'}(\zeta))$ where $z_{EE'}(\zeta) = P_{EE'} - r\zeta_E\bar{\zeta}_{E'}$

$$\xi_C\partial/\partial\bar{\zeta}(\xi^C) = \partial/\partial\bar{\zeta}(-r\bar{\zeta}_{E'})\zeta_E\nabla^{EE'}(\xi^C)\xi_C = r\xi^C\zeta^E\bar{\xi}^{E'}\nabla_{EE'}\xi_C = -r\rho.$$

From the properties of null hypersurfaces it follows that $\xi_C D\xi^C = 0$ and $\rho = \bar{\rho}$ (see [7], p. 237). The first relation gives moreover $(D\phi_B)\xi^B = D\theta - \phi_B(\xi^B\zeta_C - \zeta^B\xi_C)D\xi^C = D\theta - \epsilon\theta$. The proper choice of ξ_A on $\mathcal{N}$ can give $\epsilon = 0$ ([7]). So we have finally

$$\phi_A(P) = (2\pi i/4\pi^2)\int\int_{\zeta\in\mathbb{C}}\left[\{D\theta - (2\rho+\epsilon)\theta\}\zeta_A/r\right]r^2 d\bar{\zeta}\wedge d\zeta.$$

The formula (5) was hence recovered with $dS = ir^2 d\bar{\zeta}\wedge d\zeta$ (to have the positive orientation for Σ_3 the orientation $i\,d\bar{\zeta}\wedge d\zeta$ has to be taken in $\mathbb{C}$).

Remarks

1. If only $\Sigma_2 \subset N_P$ is given (without $\mathcal{N}$) and if ξ_A at $Q \in \Sigma_2$ is defined to represent the vector orthogonal to $T_Q\Sigma_2$ with the same

normalisation $\xi^A \zeta_A = 1$, we can have the formula

$$\phi_A(P) = 1/2\pi \int_{\Sigma_2} \left[(D\phi_A)\xi^A - 2\rho(\phi_A \xi^A) \right] (\zeta_A / r)\, dS \qquad (6)$$

without defining ξ_A outside of Σ_2 at all. The proof of such formula is the same because the most important relation $\partial/\partial\zeta(r\zeta_A)\xi^A = 0$ still holds. Another possibility is to relax this condition and to consider arbitrary (smooth) function ξ_A on Σ_2. In this case another integral formula can be obtain (if wanted) which, of course, will be more complicated and where both components of ϕ_A and $D\phi_A$ will be generally needed on Σ_2.

2. More generally, any directions $v_{AA'}$, $v_{AA'}\zeta^A\bar{\zeta}^{A'} > 0$ can be taken in the formula (4) for Σ_3. After the same procedure being done, we could obtain a lot of another integral formulas, the described procedure being the machine producing integral formulas with respect to the form of initial data hyper-surface under the consideration and with respect to the direction of derivatives chosen at corresponding points. For example, the analogue of the original Kirchhoff integral formulas, where the hypersurface is $t = 0$ and the derivatives of the field in the direction $v_{AA'} \simeq [1, 0, 0, 0]$ are considered, can be written for the field ϕ_A.

3. Up to now all fields were considered on Minkowski space. There are good reasons to believe that the procedure described above can be generalised in future to conformally flat spacetimes (Penrose's integral formula remains valid in such situation ([15], [16]). On the other hand, on conformally curved spacetimes the situation is expected to be quite different.

4. Note that the results of §3, together with the reduction to Minkowski space and next Example, gives the alternative proof of the formula (5). A special form of the Stokes theorem is used in [15], [16] for the proof of the formula (5) which gives the independence of the integral in (5) on the deformation of Σ_2 inside $\mathcal{N}$. It is just the same procedure as in Theorem 1′ on the different level (after one integration being done).

Example Let us calculate the matrix k_A^B from the proof of Theorem 2′. Taking $P = 0$, $\zeta_A = [1, \zeta]$, $\xi_A = [-\bar{\zeta}/(1 + |\zeta|^2), 1/(1 + |\zeta|^2)]$, $r = 1/(1 + |\zeta|^2)$ we get $\rho = -(1/r)\xi_C \partial/\partial\bar{\zeta}(\xi^C) = -1/(1 + |\zeta|^2)$ and (using (4) for $\phi_A(P) = \delta_A^B$) we have

$$k_A^B = 1/2\int\int 2\left[(1+|\zeta|^2)r\right]^{-1}\xi^B\zeta_A ir^2 d\bar{\zeta}\wedge d\zeta.$$

The cycle S_3 has the positive orientation if the orientation $dv\,.\,du$ is used for $\mathbb{C}$ (where $\zeta = u + iv$), so we have finally

$$k_A^B = (2/\pi)\int\int(1+u^2+v^2)^{-3}\begin{bmatrix} 1 & u-iv \\ u+iv & u^2+v^2\end{bmatrix} du\, dv.$$

which gives the unit matrix by a simple calculus.

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