ON GENERALIZED CAUCHY-RIEMANN EQUATIONS ON MANIFOLDS

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§.0. Introduction.

The definition of ∂ and $\bar{\partial}$ operators on a Riemann surface M use the splitting of the complex-valued 1-forms $\mathcal{E}^1_c = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$ and the splitting of the de Rham sequence into

$$\begin{array}{ccccc} & \overset{\partial}{\nearrow} & \mathcal{E}^{1,0} & \overset{\bar{\partial}}{\searrow} & \\ \mathcal{E}^0_c & & \oplus & & \mathcal{E}^2_c \\ & \underset{\bar{\partial}}{\searrow} & \mathcal{E}^{0,1} & \underset{\partial}{\nearrow} & \end{array} \qquad (1)$$

(see [8]). The aim of the paper is to describe a higher dimensional generalization of ∂ and $\bar{\partial}$ operators on a suitable type of manifolds such that we would be back in the classical case for n=2. The paper is only the summary of results, the detailed version of the paper with full proofs will be submitted for publication elsewhere.

A lot of work has been done already on the extension of complex analysis to higher dimensions. There are different ways how to generalize Cauchy-Riemann equations ([3],[9],[14],[24]-[26]), attempts were also made to extend hypercomplex analysis to some type of manifolds (see [15], 19 , [21]).

To indicate what type of definition we are going to suggest here let us consider a Riemannian manifold M of the dimension n. For every representation $\mathbb{E}$ of SO(n) there is the associated vector bundle E on M and the Riemannian covariant derivative $\nabla: \Gamma(E) \to \Gamma(E \otimes \Lambda^1)$, Λ^1 being the cotangent bundle. The tensor product $\mathbb{E} \otimes \Lambda^1(\mathbb{R}_n)$ can be now decomposed into the direct sum of irreducible representation spaces

$$\mathbb{E} \otimes \Lambda^1(\mathbb{R}_n) \cong \sum_{j=1}^{l} \mathbb{E}_j$$

and if we denote by π_j the projections of $E \otimes \Lambda^1$ on the associated vector bundles E_j ,then the operators $\pi_j \circ \nabla$ are examples of operators,which we are going to consider. The fact,thatsome generalizations of Cauchy-Riemann equations to $\mathbb{R}_n$ can be described using different representation spaces of the group SO(n),was explained already in [25] . The conformally invariant differential operators studied in connection with problems in physics ([6],[11],[17],[18]) are defined just in this way.

Our general definition of the splitting of a vector-valued de Rham sequence on n-dimensional manifold M will depend on the choice of the group G in consideration (in our Examples G will be SO(n) , CO(n) or their covering groups),on a choice of a representation $\mathbb{E}$ of G and on a choice of a connection on the associated vector bundle E . Let us note that while in classical case (the splitting (1)) the operators ∂ and $\bar{\partial}$ were all of the same type,their generalization to higher dimensions are usually quite different in character.

In the first part of the paper we shall state the general definition and in the second part we shall show how all examples of generalized Cauchy-Riemann equations, mentioned above,fit into the scheme.

§.1. The splitting of the vector-valued de Rham sequence.

Definition 1 :

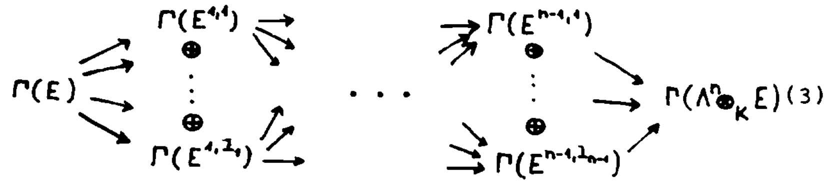
Denote by K the field $\mathbb{R}$ or $\mathbb{C}$. Consider a smooth manifold M over K ,$\dim_K M = n$ and denote by Λ^1 the corresponding cotangent space.

Let $\mathcal{B}$ be a principal fibre bundle over M with a group G. Suppose that $\mathbb{E}$ is an irreducible finite-dinensional representation of G over K and let E be the corresponding associated vector fibre bundle.Suppose further that there is a homomorphism $\rho: G \to Gl(n,K)$ such that the vector bundle,associated to the representation of G in K^n is isomorphic to Λ^1.
Then the tensor products $\Lambda^m(K^n) \otimes_K \mathbb{E}$ are also the representation spaces of G and the corresponding associated vector bundles are $\Lambda^m \otimes_K E$.

Any covariant derivative ∇ on E , $\nabla:\Gamma(E) \to \Gamma(\Lambda^1 \otimes_K E)$ can be extended in the usual way to $\nabla_j : \Gamma(\Lambda^j \otimes_K E) \to \Gamma(\Lambda^{j+1} \otimes_K E)$ and the sequence of E-valued differential forms

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(\Lambda^1 \otimes E) \xrightarrow{\nabla_1} \dots \xrightarrow{\nabla_{n-1}} \Gamma(\Lambda^n \otimes_K E) \qquad (2)$$

can be splitted as



where $\Lambda^m(K^n) \otimes_K \mathbb{E} \simeq \sum_{j=1}^{l_m} \mathbb{E}^{m,j}$ is the decomposition of the tensor product into irreducible representations of G and $E^{m,j}$ are the associated vector bundles to the representations $\mathbb{E}^{m,j}$ of G .

Remarks.

1. In many examples we have simply $G \subset Gl(n,K)$ and B is the principal bundle of the G-structure on M ,so there is no need for the homomorphism ρ . But to cover the important examples of spin bundles,we did not restrict the definition only to the case of G-structures on M .

2. There is a modification of the definition (needed in some examples),where the manifold M is real manifold,but $\mathbb{E}$ is a complex representation space of G . In this case we shall consider the complexified differntial forms Λ^*_c and the splitting of the tensor product $\Lambda^m_c \otimes_{\mathbb{C}} E$. Sometimes it is also possible to consider a complexification $\tilde{M}$ of M ,to use complex version of the definition and to restrict the operator to M after. But only the real version of the definition has the classical case of complex analysis as the special subcase (see Ex.2).

3. The operators in the splitted de Rham sequence (3) are usually quite different in character.But because each $\mathbb{E}^{m,j}$ is again an irreducible representation of G,every operator in (3) can be classified by the highest weight of the representation $\mathbb{E}^{m,j}$ and by the highest weight of the target representation $\mathbb{E}^{m+1,j}$. It would be possible to consider these individual operators as generalizations of Cauchy-Riemann equations to manifolds,but the point of view ,presented in the paper,is that it is more interesting to consider the whole splitted sequence (3). A similar splitting was already used for the description of topology of open domains $\Omega \subset \mathbb{R}_n$ (see [22]) and further results can be expected in

in this direction.

§.3. Examples.

Example 1. Regular spinor fields ([14]).

Let us take $M = \mathbb{R}^{p,q}$, $n = p+q$ with the canonical quadratic form with p positive and q negative signs.

Take $G = Spin(p,q)$, $\mathcal{B} = M \times G$, $\nabla = d$, $\rho: Spin(p,q) \to SO(p,q)$. Let $\mathbb{E} = V$ be a basic spinor representation of G (we can take e.g a minimal left ideal in the Clifford algebra $\mathcal{C}_{p,q}$).

The tensor product $\mathbb{E} \otimes_{\mathbb{R}} \Lambda^1(\mathbb{R}^{p,q})$ then splits into two pieces $\mathbb{E}_1 \oplus \mathbb{E}_2$. One of them,say $\mathbb{E}_1$,is again a basic spinor representation space (there are two of them,in even and only one in odd dimensions). It is sufficient to consider the splitting of 1-forms:

$$\Gamma(E) \xrightarrow{d} \Gamma(E \otimes_{\mathbb{R}} \Lambda^1) \begin{array}{l} \xrightarrow{\pi_1} \Gamma(E_1) \\ \qquad \oplus \\ \xrightarrow{\pi_2} \Gamma(E_2) \end{array}$$

and the equation $(\pi_1 \circ d)\psi = 0$, $\psi: \mathbb{R}^{p,q} \to \mathbb{E}$ is just the generalized Cauchy-Riemann equation for regular spinor fields,described in [14]. The equation $(\pi_1 \circ d)\, \psi = 0$ is of elliptic type for $p=0$ or $q=0$,whil for other cases it is hyperbolic or ultrahyperbolic system.

Example 2. Complex analysis.

Take $G = SO(2)$ in the special case $n = 2$ of Example 1. A manifold M ($\dim_{\mathbb{R}} M = 2$) with G-structure and the corresponding principal fibre bundle $\mathcal{B}$ can be identified with the complex manifold M (with the choosen Hermitian structure on the tangent bundle). If we take $\mathbb{E} = \mathbb{C}$ with the trivial representation of G ,then the tensor product $\Lambda^1(\mathbb{R}_2) \otimes_{\mathbb{R}} \mathbb{C}$ splits into two irreducible pieces (each of real dimension 2). If we moreover set $\nabla = d$,then the sequence (2) from the Def.1 is just the de Rham sequence $\Lambda^0_c \xrightarrow{d} \Lambda^1_c \xrightarrow{d} \Lambda^2_c$ and the splitting (3),given by Def.1.,coincide with the standard splitting (1). The classical Cauchy-Riemann equatioms on complex manifold are hence recovered.

Example 3. Clifford analysis ([3] , [5]).

A. Let us take $M = \mathbb{R}^{0,n}$, $G = Spin(n)$, $\mathcal{B} = M \times G$, $\mathbb{E} = C_n$, where C_n is the Clifford algebra for $\mathbb{R}^{0,n}$ (the negative definite norm). The group G can be imbedded into C_n in the standard way, so $\mathbb{E}$ is the representation space for G. But C_n is not irreducible space,it can be decomposed into a sum of real spinor representations. It is easy to see it in the matrix representation of Clifford algebras ([12], p.148). The spinor subspaces are just columns or half-columns of the corresponding matrices. The table of real spinor representations with their dimensions is given in ([12],p.193).

Because all irreducible pieces in a decomposition $C_n = E^{(1)}\oplus \cdots \oplus E^{(1)}$ are the same (i.e. isomorphic) representation spaces of G , we shall apply the Def.1. to each piece with the same result as in Example 1 :

$$\Gamma(E^{(j)}) \xrightarrow{d} \Gamma(E^{(j)}\otimes_{\mathbb{R}}\Lambda^1) \begin{array}{l} \xrightarrow{\pi_1} \Gamma(E_1^{(j)}) \\ \qquad \oplus \\ \xrightarrow{\pi_2} \Gamma(E_2^{(j)}) \end{array}$$

and we shall add them together after

$$\Gamma(C_n) \simeq \Gamma(E^{(1)}\oplus\cdots\oplus E^{(1)}) \begin{array}{l} \xrightarrow{\pi_1\circ d} \Gamma(E_1^{(1)}\oplus\cdots\oplus E_1^{(1)}) \simeq \Gamma(C_n) \\ \xrightarrow{\pi_2\circ d} \Gamma(E_2^{(1)}\oplus\cdots\oplus E_2^{(1)}) \end{array}$$

Then the operator $\pi_1\circ d$ can be identified with the operator

$$\Big(\sum_{i=1}^{n} e_i\partial_i\Big)\psi \quad \text{for mappings} \quad \psi:\mathbb{R}^{0,n}\longrightarrow C_n$$

described in [5].

B. A lot of function theory and transform analysis is already known for so called monogenic functions ψ on $\mathbb{R}_{n+1}$ with values in C_n (see [3]). They are solutions of generalized Cauchy-Riemann equations $\big(\sum_{i=0}^{n} e_i\,\partial_i\big)\psi = 0$,where $e_1,\ldots,e_n$ are generators of C_n and $e_0 = 1$.

Let us take in Definition 1 $M = \mathbb{R}_{n+1}$, $G = Spin(n+1)$, $B = M \times G$, $E = C_n$, $\nabla = d$ and $\rho:G \to SO(n+1)$. Clifford algebra C_n is the representation space of G ,because $Spin(n+1) \subset C^{+}_{n+1} \simeq C_n$. where $C^{+}_{n+1}(C^{-}_{n+1})$ means the even (odd) part of C_{n+1} (see [12] ,p.185) Then the use of Def.1 in the same way as in Ex.3.A.give us just the equation for monogenic functions. To see it,let us consider Ex.3.A. in dimension n+1.

The procedure of Example 3.A. give us the equation

$$\Big(\sum_{i=0}^{n} \delta_i\,\partial_i\Big)\psi = 0 \tag{4}$$

where $\delta_0,\ldots,\delta_n$ are generators of C_{n+1} and ψ is a map from M into C_{n+1} . We shall identify C_n with C^{+}_{n+1} by the correspondence

$$e_i \sim -\delta_0\cdot\delta_i \quad , \; i = 1,\ldots,n \; .$$

The maps $\psi : \mathbb{R}_{n+1} \longrightarrow C_n$ are then special cases of maps from $\mathbb{R}_{n+1}$ into C_{n+1} and the equation (4) reduces for them (after multiplica tion by δ_0) to the equation

$$(\sum_{i=0}^{n} e_i \, \partial_i)\psi = 0 \tag{5}$$

where $e_0 = \delta_0 \cdot \delta_0 = 1$.

<u>Example 4</u>. Fueter's regular functions ([26]).

Fueter and his coworkers studied quaternionic analysis in 30's and 40's . This case is in fact the special case of example 3.B.for $n = 3$. In this case $M = \mathbb{R}_4 \simeq \mathbb{H}$ and $\mathbb{E} = C_3 \simeq \mathbb{H} \oplus \mathbb{H}$. So it is suffi cient to split the value of $\psi = (\psi_1, \psi_2)$ and to consider only a half of it . The equation (5) then reduces to the classical Fueter equa- tion.

<u>Example 5.</u> Massless fields on Minkowski space ([2] , [7]).

Take $M = \mathbb{R}^{1,3}$ (Minkowski space) , G = Lorentz group. There ar two basic spinor representations V and V' of G (over $\mathbb{C}$) and we shall denote $E^{j,l} = S^j V \otimes S^l V'$,where S^n denotes the n-th symmetric tensor product of the corresponding vector space. Take $B = M \times G$, $\nabla = d$, $\rho = id$. We shall use Remark 2,after the Def.1 and we shall complexify the cotangent space to Λ^1_c . Then

$$E^{j,l} \otimes_{\mathbb{C}} \Lambda^1_c \simeq E^{j+1,l+1} \oplus E^{j-1,l+1} \oplus E^{j+1,l-1} \oplus E^{j-1,l-1}$$

(if j=0 or l=0,then the corresponding spaces are missing). So we ha ve the splitting

$$\Gamma(E^{j,l}) \xrightarrow{d} \Gamma(\Lambda^1_c \otimes_{\mathbb{C}} E^{j,l}) \begin{array}{l} \xrightarrow{\pi_1} \Gamma(E^{j+1,l+1}) \\ \xrightarrow{\pi_2} \Gamma(E^{j-1,l+1}) \\ \xrightarrow{\pi_3} \Gamma(E^{j+1,l-1}) \\ \xrightarrow{\pi_4} \Gamma(E^{j-1,l-1}) \end{array}$$

The four differential operators obtained in such a way are just the equations,described by Garding ([7]). In the case j = 0 or l = 0 there are only two pieces in the decomposition. They are usually called massless field equation and twistor equation for spin j/2 (resp. l/2) fields ([2])

<u>Remark</u>.

Massless field. are very often considered on complexified Min- kowski space $\mathbb{C}M$ and restricted after either to (real) Minkowski space or to Euclidean space. In this case we can use the complex ca se of the Def.1.and the massless fields described above are restric tions of this complex case to (real) Minkowski space.

Example 6. Complexified Clifford analysis ([20]).

It is the complexified version of the example 3. Take $M = \mathbb{C}_n$, $G = SO(n,\mathbb{C})$, $E = C_n^c$ (complex Clifford algebra), $B = M \times G$, $\nabla = d$, $\rho = id$. The Clifford algebra C_n^c is again reducible and it consists of many copies of basic spinor representations of G. Applying the same procedure as in Example 3.A,B., we shall obtain by splitting of 1-forms the equations for regular mappings in complex Clifford analysis (see [20]).

Remark.

The case $n = 4$ is especially interesting because of the connection to mathematical physics. The complex Clifford algebra can be decomposed into spinor spaces and we shall be back in the situation of Example 5. If we split the algebra as $C_4^c \simeq \mathbb{C}(2) \oplus \mathbb{C}(2)$, we shall be in the case of complexified Fueter equation ([16], [23]). There is a nice connection between Cauchy integral formula for such mappings and various integral formulas in mathematical physics ([23]).

Example 7. Stein & Weiss generalization of C-R equations ([25]).

The paper by Stein and Weiss introduced generalized Cauchy--Riemann equations on $\mathbb{R}_n$ using Spin(n) representations. It is again the case where the space $M = \mathbb{R}_n$ is real, but representation spaces $\mathbb{E}$ are complex (see Remark 2 after Def.1).

So $M = \mathbb{R}_n$, $G = Spin(n)$, $\mathbb{E}$ is any irreducible representation of G, $B = M \times G$, $\nabla = d$, $\rho: Spin(n) \to SO(n)$.

There is the exceptional irreducible piece in the product $\mathbb{E} \otimes_{\mathbb{C}} \Lambda_c^1$ called the Cartan product of $\mathbb{E}$ and Λ_c^1. It is characterized by the fact that its highest weight is the sum of the highest weights of $\mathbb{E}$ and Λ_c^1. Let us denote it by $\mathbb{E}_1$, so $\mathbb{E} \otimes_{\mathbb{C}} \Lambda_c^1 \simeq \mathbb{E}_1 \oplus \mathbb{E}_2$, where $\mathbb{E}_2$ can now be a reducible representation. Denoting again π_1, π_2 the projection to $\mathbb{E}_1$, $\mathbb{E}_2$, we can write Stein and Weiss equations as $(\pi_2 \circ \nabla)\,\psi = 0$. They are reducible in general and contain as special cases Euclidean form of Dirac equation (if E is a basic spinor representation) and the Hodge operator $d + \delta$ (if $\mathbb{E} = \Lambda_c^r$, $r = 0,\dots,n$). Both these special cases are well studied on manifolds ([10]).

Example 8. Conformally invariant differential equations ([18], [6])

Suppose that M is an oriented manifold with a conformal structure, dim $M = n$ (i.e. $G = CO(n)$). Let B be the corresponding principal fiber bundle, $\rho = id$.

Representation spaces of G are classified by a representation space $\mathbb{E}$ of $SO(n)$ together with a conformal weight w . Let us choose now a Riemannian metric within the conformal structure and denote by ∇ the corresponding covariant derivative.

The splitting of the map $\nabla: \Gamma(E) \to \Gamma(\Lambda^1 \otimes E)$ into irreducible pieces is now given by the procedure of Def.1. The corresponding operators depend generally on the choice of the Riemannian connection on M. It was shown in [6], [18] that for every representation space $\mathbb{E}$ of $SO(n)$ and for every $j = 1,\dots,l$ there is just one conformal weight w such that the operator $\pi_j \circ \nabla$ is conformally invariant,i.e. it does not depend on the choice of the Riemannian structure within the given conformal class.

Remarks.

1. The standard procedure ([6], [11], [17]) is to consider the representation spaces $\mathbb{E}$ over $\mathbb{C}$, to complexify the cotangent space and to split $\Lambda^1_c \otimes_{\mathbb{C}} \mathbb{E}$.
2. If a spinor representation is used,the group $CO(n)$ has to replaced by its covering group and the operators are globally well--defined only on a spin manifold M .

Example 9. Generaliz d C-R equation on manifolds.

Let us now discuss how two cases of Cauchy-Riemann equations on special types of manifolds,described in [15], [19]and[21],fit into the scheme.

A. Let M be an oriented 4-manifolds with a conformal structure Denote $\tilde{G} = CO(4)$ and $G = Spin(n)\cdot R_+$ (the covering group of $\tilde{G}$). Let $\rho: G \to \tilde{G}$ be the covering map. Define ∇ as a Riemannian connection within the conformal structure on M . Suppose further that the principal fiber bundle $\tilde{B}$ for $\tilde{G}$-structure on M can be lifted to a principal fiber bundle B together with the bundle map $\bar{\sigma}: B \to \tilde{B}$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\bar{\sigma}} & \tilde{B} \\ g \downarrow & \bar{\sigma} & \downarrow \rho(g) \\ B & \longrightarrow & \tilde{B} \end{array}$$

commutes for every $g \in G$.

The action of $Spin(4) \simeq Sp(1) \times Sp(1)$ on $R_4 \simeq \mathbb{H}$ can be described using quaternions as

$$(a,b) \in Sp(1) \times Sp(1) \longmapsto (a,b)q = aqb \ , \ q \in \mathbb{H}.$$

Every representation of G is characterized by a $Spin(4)$ representation and by a conformal weight w . We shall consider the representation of G on $\mathbb{E} = \mathbb{H}$ given by

$\tau:(a,b)\in Spin(4) \longmapsto \tau(a,b)q = b^{-1}q$, $q\in \mathbb{H}$
and by the conformal weight $w = 3/2$. Then the splitting

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(\Lambda^1 \otimes_{\mathbb{R}} E) \begin{array}{l} \xrightarrow{\pi_1} \Gamma(E_1) \\ \quad\oplus \\ \xrightarrow{\pi_2} \Gamma(E_2) \end{array}$$

correspond to $\mathbb{E} \otimes_{\mathbb{R}} \Lambda^1(\mathbb{E}) = \mathbb{E}_1 \oplus \mathbb{E}_2$, where $\mathbb{E}_1 \simeq \mathbb{H}$ is again a basic spinor representation of Spin(4). The operator $\pi_1\circ\nabla$ is conformally invariant, while $\pi_2\circ\nabla$ is not (it needs another conformal weight [6]). Hence for every such a manifold M we have associated two vector bundles E and E_1 over M and the differential operator $\pi_1\circ\nabla$ from $\Gamma(E)$ to $\Gamma(E_1)$. This is just the more detailed description of one conformally invariant operator -Dirac operator- discussed in Example 8. Now in the case that the conformal structure is integrable, we can find a coordinate covering such that the transition functions are of the type $y = (ax+b)(cx+d)^{-1}$, $x,y,a,b,c,d \in \mathbb{H}$ (see [13]). In this case there is a way how to define the bundle E and the equation $(\pi_1\circ\nabla)\,\psi = 0$ directly by patching together flat pieces. It was done in [15] . In local quaternionic coordinates the equation reduces to Fueter equation.

Remark.

An extension of Markl's construction to complex Clifford case was described in [19] . A further study is needed to see what general structure corresponds to this case.

B. Let M be a Riemannian manifold (dim M = n). with an exterior structure ,given by the Weingarten map. Take G = SO(n), $\mathcal{B}$ the corresponding principal fibre bundle , $\mathbb{E} = C_n$ (then E is usually called the Clifford bundle). A special connection ∇ was defined on E using the Riemannian connection and the Weingarten map in [21] . The splitting $\mathbb{E} \otimes_{\mathbb{R}} \Lambda^1 = \mathbb{E}_1 \oplus \mathbb{E}_2$ being the same as in Ex.3.A., we shall get the operator $\pi_1\circ\nabla : \Gamma(\mathbb{E}) \longrightarrow \Gamma(E_1)$,which was presented in [21] as the generalization of the spherical Cauchy-Riemann operator Γ ([3]) .

Example 10 . Quaternoonic-valued differential forms on $\mathbb{H}$ ([24]).

Take $M = \mathbb{R}_4 \simeq H$, G = Spin(4), $\mathcal{B}$ = M x G, $\nabla := d$,
ρ:Spin(4) $\longrightarrow$ SO(4). We can identify $\mathbb{E} = \mathbb{H}$ with the spinor representation by

$\tau:(a,b) \in Sp(1) \times Sp(1) \simeq Spin(4) \longmapsto \tau(a,b)\, x = a\cdot x$, $x\in \mathbb{H}$.

Let us denote this representation by V and the other spinor representation space by V' . We shall denote (as in Ex.5)

$\mathbb{E}^{j,l} = S^j V \otimes_{\mathbb{C}} S^l V'$. Note that $\mathbb{E} = \mathbb{E}^{1,0}$. The spaces $\mathbb{E}^{j,l}$ are complex vector spaces,but they are real spin modules for j+l even and quaternionic ones for j+l odd (see [19]). So

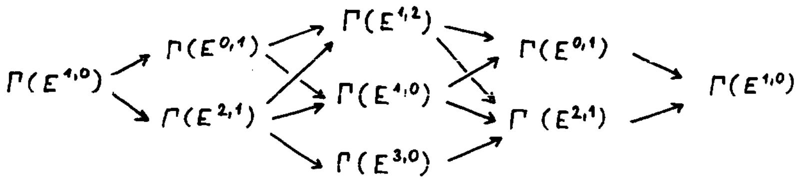
So for j+l even there is a real subspace $r\mathbb{E}^{j,l} \subset \mathbb{E}^{j,l}$,which is the irreducible (over $\mathbb{R}$) representation space of SO(4). Note that $r\mathbb{E}^{j,l} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{E}^{j,l}$ (see [1] or [12]).We have the following isomorphisms of representation spaces

$$\Lambda^1(\mathbb{R}_4) \simeq r\mathbb{E}^{1,1} \quad (\text{i.e. } \Lambda^1_c(\mathbb{R}_4) \simeq \mathbb{E}^{1,1})$$
$$\Lambda^2(\mathbb{R}_4) \simeq r\mathbb{E}^{2,0} + r\mathbb{E}^{0,2} \quad (\text{the Hodge decomposition})$$
$$\Lambda^3(\mathbb{R}_4) \simeq \Lambda^1(\mathbb{R}_4) \ , \quad \Lambda^4(\mathbb{R}_4) \simeq \Lambda^0(\mathbb{R}_4) \simeq \mathbb{R}$$

It then leads to the splittings

$$\mathbb{E}^{1,0} \otimes_{\mathbb{R}} \Lambda^1 \simeq \mathbb{E}^{0,1} \oplus \mathbb{E}^{1,2} \ , \ \dim_{\mathbb{R}} \mathbb{E}^{0,1} = 4, \dim_{\mathbb{R}} \mathbb{E}^{2,1} = 12,$$
$$\mathbb{E}^{1,0} \otimes_{\mathbb{R}} \Lambda^1 \simeq \mathbb{E}^{1,0} \otimes_{\mathbb{R}} (r\mathbb{E}^{2,0} \oplus r\mathbb{E}^{0,2}) \simeq \mathbb{E}^{3,0} \oplus \mathbb{E}^{1,0} \oplus \mathbb{E}^{1,2}$$
$$\dim_{\mathbb{R}} \mathbb{E}^{3,0} = 8 \ , \ \dim_{\mathbb{R}} \mathbb{E}^{1,0} = 4 \ , \ \dim_{\mathbb{R}} \mathbb{E}^{1,2} = 12$$
$$\mathbb{E}^{1,0} \otimes_{\mathbb{R}} \Lambda^3 \simeq \mathbb{E}^{0,1} \oplus \mathbb{E}^{1,2}$$

The splitting (3) in the Def.1. has then the form



(6)

It can be compared with the splitting of $\mathbb{H}$-valued forms described in [24] using quaternionic coordinates. The splitting of 1-forms and 3-forms coincide,while 2-forms were splitted in [24] only into two pieces $\mathbb{E}^{1,2} \oplus (\mathbb{E}^{1,0} \oplus \mathbb{E}^{3,0})$ (this splitting correspond just to the splitting into selfdual and antiselfdual forms). So the point of view presented here gives the refinement of the splitting in [24]. Moreover ,the same splitting can be considered (after a choice of a connection on E) on any spin 4-manifold.

Remarks.

1. Note that all operators in (6) are massless field operators described by Garding [7] ,considered , of course,on Euclidean space-time

2. The splitting of Clifford-valued forms on $\mathbb{R}_{n+1}$ described in [22] does not fit into the scheme,so it cannot be extended to manifolds using described methods.

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