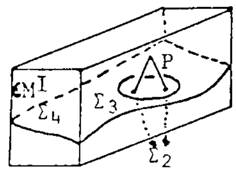
Boundary value type & initial value type integral formulas for massless fields.

Even if the standard twistor integral formulas are well-designed to describe solutions of massless field equations, they are not integral formulas giving a solution to initial value (resp. boundary value) problem. Such formulas were described in [1] for real Minkowski space M^I. It will be shown here that there are two possible types of formulas in $\mathbb{C}M^I$ – either an initial value type one, which is an 'analytic continuation' of Penrose's one from M^I, or a boundary value type one, which is an 'analytic continuation' of Cauchy-type integral formula from Euclidean space-time.

To have a picture of the situation, take $\Sigma_4 \subset \mathbb{C}M^I$ such that Σ_4 intersects each α-plane in the unique point. The value of a massless field in a point P can be reconstructed from (a suitable combination of) values and first derivatives of the field on the intersection Σ_2 of the complex null cone N_P (with the vertex at P) with Σ_4 (an initial value type formula). This formula can be also used to describe the inverse Penrose transform, it reduces for $\Sigma_4 = \Sigma$ (a negative definite real slice of M^+) to formulas in [2]. On the other hand, given values of the field on $\Sigma_3 \subset \Sigma_4$, there is an integral formula expressing the value of the field at P for every P, for which the corresponding intersection Σ_2 lies 'inside' of Σ_3 (a boundary value type formula). The transition from boundary to initial value type formulas is done by the integration with respect to a (suitably chosen) variable using the residue theorem, the singularity being just on the null cone.



The boundary value type formula.

Let us denote $DQ_{AA'} := \frac{1}{2}\begin{bmatrix} dQ_{00'}\wedge dQ_{01'}\wedge dQ_{10'}; & -dQ_{00'}\wedge dQ_{10'}\wedge dQ_{11'} \\ dQ_{00'}\wedge dQ_{01'}\wedge dQ_{11'}; & -dQ_{01'}\wedge dQ_{10'}\wedge dQ_{11'} \end{bmatrix}$. Consider a point $P^{AA'} \in \mathbb{C}M^I$, the 4-dimensional ball $U_\rho(P)$, lying in the Euclidean slice of $\mathbb{C}M^I$ through P and its boundary $S_3 = \partial U_\rho(P)$. Then we have

Theorem 1 (Cauchy integral formula):

Let $\nabla_{AA'}\psi^{A'..E'} = 0$ in $\Omega \subset \mathbb{C}M^I$, $P \in \Omega$. Let $\Sigma_3 \sim S_3$ (i.e. belongs to the same class of homology) in $\Omega \setminus \mathbb{C}N_P$. Then (spin n/2 case)

$$\psi^{A'..E'}(P) = 1/2\pi^2 \int_{\Sigma_3} \psi^{F'..H'}\, DQ_{(F(F'}(Q-P)_{GG'}..(Q-P)_{H)H')}\ \frac{(Q-P)^{FA'}...(Q-P)^{HE'}}{|Q-P|^{2n+2}} .$$

For spin 0 case ($\Box\psi = 0$ in Ω) we have

$$\psi(P) = 1/4\pi^2 \int_{\Sigma_3} \left[\psi(Q)\, \overleftrightarrow{\nabla}^{AA'} \frac{1}{|Q-P|^2}\right] DQ_{AA'} .$$

Proof: It is possible to use the induction on n . For $n=1$ (and $n=0$) the assertion is the consequence of the standard Stokes theorem (the integrand is a closed 3-form) and of the fact that $1/2\pi^2 \int DQ_{AA'}\ (Q-P)^{AB'}/|Q-P|^4 = \epsilon_{A'}{}^{B'}$ (see [3]).

$$\int_{\Sigma_3} \psi^{C'D'} DQ_{(C(C'}(Q-P)_{D)D')} \quad (Q-P)^{CA'}(Q-P)^{DB'}/|Q-P|^6 =$$

$$= \tfrac{1}{2} \int_{\Sigma_3} \psi^{C'B'} DQ_{CC'}(Q-P)^{CA'}/|Q-P|^4 + \psi^{A'D'} DQ_{DD'}(Q-P)^{DB'}/|Q-P|^4$$

The contours of integration.

Some examples of Σ_3 homological to S_3 are needed.

Ex.1: If S_3' is a sphere in an Euclidean slice E_4 and if $\mathbb{C}N_p \cap E_4$ is contained in the interior of S_3', then $S_3' \sim S_3$ in $\mathbb{C}M^I \setminus \mathbb{C}N_p$.

Ex.2: Let ξ^A be a (-1)-homogeneous function of $\pi^{A'} \in \mathbb{C}_2 \setminus \{0\}$ and define

$\Sigma_2 = \{P^{AA'} - \xi^A \pi^{A'} \mid \pi^{A'} \in \mathbb{C}_2 \setminus \{0\}\}$. For the purposes of integration it is sufficient to consider $\pi^{A'} := [1, \pi]$, $\pi \in \mathbb{C}$. Let us choose further spinor fields $o^A, \iota^{A'}$ on Σ_2 such that $\xi_A o^A = \pi_{A'} \iota^{A'} = 1$ and that the orthogonality conditions $\partial/\partial\pi(\xi_A \pi_{A'}) o^A \iota^{A'} = \partial/\partial\bar{\pi}(\xi_A \pi_{A'}) o^A \iota^{A'} = 0$ hold on Σ_2. Let us denote further

$\Sigma_3 = \{P^{AA'} - \xi^A \pi^{A'} + z o^A \iota^{A'} \mid \pi^{A'} \in \mathbb{C}_2 \setminus \{0\},\ z \in \mathbb{C},\ |z| = \varepsilon\}$, $\varepsilon > 0$ fixed.

Then $\Sigma_3 \sim S_3$ in $\mathbb{C}M^I \setminus \mathbb{C}N_p$ (it can be shown by a simple homotopy argument that $\Sigma_3 \sim \Sigma_3''(1)$ from [3])..

The initial value type formula.

Theorem 2: Let $\psi^{A'..E'}$ $(n \geq 0)$ satisfy the massless field equation in M^I. Then (with notation as in Ex.2)

$$(*) \quad \psi^{A'..E'}(P) = (-1)^{n+1}/2\pi \int \Big[(\mathfrak{D}\psi^{F'..H'})\iota_{F'}..\iota_{H'}.a_1 + (n+1)(\psi^{F'..H'}\iota_{F'}..\iota_{H'}).a_2 \Big] .\pi^{A'}..\pi^{E'} .i d\pi \wedge d\bar{\pi} \,..$$

where $\mathfrak{D} = o^A \iota^{A'} \nabla_{AA'}$, $a_1 = \xi^A \partial/\partial\bar{\pi}\, \xi_A$, $a_2 = \iota^{A'} \partial/\partial\bar{\pi}\, \iota_{A'}$.

Proof:

Taking Σ_3 as in Ex.2, the formula (*) is obtained by using Theorem 1 and the residue formula in the variable z . The computation is similar as in [3] . Further information needed is:

- the orthogonality conditions imply the relation

$$\iota^{A'} \partial/\partial\bar{\pi}(\iota_{A'}) = \partial/\partial\pi(\xi_A)\partial/\partial\bar{\pi}(o^A) - \partial/\partial\bar{\pi}(\xi_A)\partial/\partial\pi(o^A),$$

- with this we have

$$DQ_{BB'} = \tfrac{1}{2}(a_1 + 2za_2)\, o_B \iota_{B'}\, dz \wedge d\pi \wedge d\bar{\pi} \,.$$

This information is sufficient for the case n=o and n=1, but for higher spins another formula is needed. Such formula can be used also for the inductive proof of (*) (starting with n=o case), which can be of an independent interest. Let us show it on the case (spin 0 → spin ½).

Suppose that $(*)$ holds for $n=0$. If $\psi^{A'}$ is a spin $\frac{1}{2}$ massless field, then using $n=0$ formula we have $\psi^{A'} = -1/2\pi\int(\mathfrak{D}\psi^{A'}.a_1+\psi^{A'}.a_2)\,i\,d\pi\wedge d\bar{\pi}$.
Inserting $\epsilon_{B'}{}^{A'} = \pi_{B'}\iota^{A'} - \iota_{B'}\pi^{A'}$ we have

$$\psi^{A'}=1/2\pi\int\{[(\mathfrak{D}\psi^{B'})\iota_{B'}.a_1+\psi^{B'}\iota_{B'}.a_2]\pi^{A'} - [(\mathfrak{D}\psi^{B'})\pi_{B'}.a_1+\psi^{B'}\pi_{B'}.a_2]\iota^{A'}\}\,i\,d\pi\wedge d\bar{\pi} .$$

Now adding the integral of an exact form

$$(**)\quad -d(\psi^{B'}\iota_{B'}\iota^{A'}d\pi) = [a_1(\mathfrak{D}\psi^{B'})\pi_{B'}\iota^{A'} + a_2\psi^{B'}\pi_{B'}\iota^{A'} + a_2\psi^{B'}\iota_{B'}\pi^{A'}]\,d\pi\wedge d\bar{\pi} .$$

we immediately obtain the formula $(*)$ for $n=1$. The same scheme works also for the induction step from n to $n+1$. For the relation $(**)$ we have need of the property that the massless field equation for $\psi^{A'}$ is equivalent with the property $\nabla_{AA'}\psi_{B'} = \nabla_{A(A'}\psi_{B')}$ (see [1]).

Remarks:

1. In the case $\xi^A = r\bar{\pi}^A$, $\pi_{A'}\iota^{A'}=1$, $o^A=1/r.\bar{\iota}^A$; $r>0$, $r=r(\pi,\bar{\pi})$, the formula $(*)$ reduces to the (primed version of) Penrose's formula from [1] . In this case we have $a_1=-r^2$; $a_2=-\iota^{C'}\bar{\iota}^{E}\pi^{E'}\nabla_{EE'}(\iota_{C'}).r$; $ir^2 d\pi\wedge d\bar{\pi} = d\mathcal{S}$, and $\mathfrak{D}=1/r\bar{\iota}^{E}\iota^{E'}\nabla_{EE'}$

The term $(\mathfrak{D}\psi^{A'})\iota_{A'}$ can be written as $\tilde{\mathfrak{D}}(\psi^{A'}\iota_{A'})$ with $\tilde{\mathfrak{D}}$ equal to $\mathfrak{D}$ minus a spin coefficient (as in [3]), for this case of course the field $\iota_{A'}$ has to be extended to a neighbourhood of Σ_2 in a suitable way.

2. The explicite formula from [2] for the inverse Penrose transform can be obtained (with notations from [2]) taking $\iota^{A'} = 1/t.\ t^{AA'}\bar{\pi}_A$, ξ^A (as in [2]) depends on $\pi^{A'}$ in such a way that Σ_2 is just the intersection of N_p with the negative definite slice Σ of M^+. Then $a_2=-1/t^2$, $-a_1 o^E d\bar{\pi}\wedge d\pi = dr^{EE'}\iota_{E'}\wedge d\pi$ and the formula $(*)$ reduces to formulas (12) and (29) from [2] .

3. To have a local version of Theorem 2, we have to suppose that the massless field equation is satisfied in a neighbourhood of (a suitable part of) complex null cone $\mathbb{C}N_p$.

References:

[1] R.Penrose: Null hypersurface initial data for classical fields... , GRG 12 (1980), 225-264

[2] N.M.J.Woodhouse: Twistor cohomology without sheaves, Adv.Twistor Th. (2.3.)

[3] V. Souček: Complex-quaternionic analysis applied to spin-$\frac{1}{2}$ massless fields to be published in Complex variables, theory and applications