Monogenic differential forms.

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Abstract

In this paper Clifford- or spinor-valued differential forms are studied. It is shown that it is possible to define monogenic differential forms as solutions of Spin-invariant systems of differential equations in such a way that such monogenic forms, defined on a domain $\Omega \subset R^n$, reflect faithfully the topology of Ω and that the Cauchy theorem holds for them.

1 Introduction.

Starting with the pioneering papers by G.C.Moisil and R.Fueter ([6, 11]) in 30's the effort of many people (see e.g.[3, 1, 8, 9, 10, 12, 17]) established the (Riemannian version of) Dirac operator as the appropriate generalization of Cauchy-Riemann equations to higher dimensions and a comprehensive function theory was already develcped for $\mathcal{C}_m$-valued solutions of the Dirac equation (usually called monogenic functions) ([1]). The next natural topic - the question what is the best generalization of holomorphic forms to higher dimensions - has already come through a longer evolution. Such generalizations can be based on different points of view ([14, 13, 15, 16]).

The generalization presented in the paper is based on three basic requirements, we believe that they are indispensable for a good answer to the question how to define in the best way the notion of a monogenic differential form.

The first and the most important one is certainly the condition of Spin(m) invariance of the scheme. It is the property which make possible to generalize the monogenic forms to Spin-manifolds.

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The second requirement is topological. In the holomorphic case (n=2) it is possible to describe the topological information on the domain (such as, for example, the number of holes in it) using holomorphic forms and the homology of the sequence

$$0 \to C \to \Gamma(\Omega, \mathcal{O}) \xrightarrow{d} \Gamma(\Omega, \mathcal{O}^{1,0}) \to 0, \tag{1}$$

where $\Gamma(\Omega, \mathcal{O})$ denotes holomorphic functions on $\Omega \subset R^2$ and $\Gamma(\Omega, \mathcal{O}^{1,0})$ denotes holomorphic 1-forms on Ω. So, for example,

$$H_1(\Omega, C) \cong \Gamma(\Omega, \mathcal{O}^{1,0}) / \ Im\, d.$$

It then reduces to the question how many holomorphic functions on Ω have no primitives on Ω. We want to define the spaces $\mathcal{M}^k$ of monogenic $\mathcal{C}_m$-valued k-forms on $\Omega \subset R_m$ in such a way that the homology of the sequence

$$0 \to \mathcal{C}_m \to \Gamma(\Omega, \mathcal{M}^0) \xrightarrow{d} \Gamma(\Omega, \mathcal{M}^1) \ldots \xrightarrow{d} \Gamma(\Omega, \mathcal{M}^{m-1}) \to 0, \tag{2}$$

at the k-th place concides with the homology spaces $H_k(\Omega, \mathcal{C}_m)$.

Finally, the third condition is coming from the function theory point of view. The standard function theory for monogenic functions ([1]) uses not only monogenic functions, but also monogenic (m-1)-forms. They are the forms $\omega = d\sigma f$, where f is monogenic and $d\sigma = \sum_{i=1}^{m} (-1)^{i+1} e_i d\hat{x}_i$ (where the hat indicates that i-th coordinate is missing in the product). The Cauchy theorem and its further consequences are based on the fact that if f and $\omega = d\sigma g$ are monogenic, then $\tau = f \wedge \omega$ is closed. The duality theory, described in [4], is also based on this property. It is certainly desirable to define monogenic forms in such a way that a similar property holds.

We want to describe now a possibility how to define monogenic forms in such a way that all requirements, described above, are satisfied. In all previous discussions of such generalizations only a few of them were Spin(m) invariant (some examples in [15]), some of them were able to describe homology of the domain ([14, 15, 16]). The monogenic forms, defined in [16] were very close to the answer, they were invariant, they expressed the homology of the domain in a slightly different version and the Cauchy theorem was valid not for the wedge product of forms, but for another coupling between forms. In this sense the paper gives the answer wanted for some time already and it offers a net solution to the problem.

This paper is designed just to announce and explain the results, we shall give no detailed proofs. The detailed (and extended) version of the theory will be published later. The paper is organised as follows. In the first section we shall describe the splitting of $\mathcal{C}_m$-valued k-forms into smaller, Spin(m) invariant, pieces. Then, in the second section, we shall build two complementary subspaces in the space of all k-forms, using these invariant pieces. The analogy with the definition of holomorphic forms will suggest how to use these two parts for the definition of monogenic forms. In the last section we shall describe the properties of monogenic forms.

2 Clifford- and spinor-valued forms.

Let us first summarize some basic facts on differential forms on domains in R^m with values in the Clifford algebra $\mathcal{C}_m$ (associated to R^m with the standard negative definite form). They are just maps from $\Omega \subset R^m$ into $\mathcal{C}_m \otimes (\wedge^\bullet R^m)$. Denote such maps by

$$\mathcal{E}^\bullet(\Omega, \overset{\mathcal{C}_m}{\mathcal{E}_m}) = \oplus\, \mathcal{E}^k(\Omega, \mathcal{C}_m).$$

We shall consider spinor-valued forms as well. It is well-known that in even dimensions there are two basic irreducible (complex) spinor representations S^+, S^-, while in odd dimensions there is just one spinor space S. We shall denote either of these spinor representations in any dimension simply by S. The space of spinor-valued forms will be denoted by $\mathcal{E}^\bullet(\Omega, S)$. To consider spinor-valued forms is not very different from considering $\mathcal{C}_m$-valued forms, because $\mathcal{C}_m$ can be decomposed (as the Spin(m)-representation) into a sum of spinor spaces. We shall write most of formulas for forms with $\mathcal{C}_m$-coefficients, to translate everything into spinor-valued case, it is sufficient to multiply everything from the right (resp. left) by a suitable constant spinor.

The simplest examples are 0-forms (i.e. functions) and special (m-1)-forms of the type $\omega = d\sigma.f$, where $f \in \mathcal{E}^0$ and $d\sigma = \sum_{i=1}^m (-1)^{i+1} e_i d\hat{x}_i$. To describe the special form $d\sigma$ in an invariant way we shall use the the contraction $t\rfloor\omega$ of a form $\omega = \sum_{|A|=k} \omega_A dx_A \in \mathcal{E}^k$ and ~~the~~ a $\mathcal{C}_m$-valued vector field $t = \sum_1^m t_j \partial_{x_j}, t_j \in \mathcal{C}_m$ given by

$$t\rfloor\omega = \sum_{j,A} t_j F_A(\partial_{x_j}\rfloor dx_A).$$

Similarly we define

$$F\lfloor t = \sum_{j,A} F_A t_j (-1)^{k-1} (\partial_{x_j} \rfloor dx_A).$$

Using such notation, we find that

$$d\sigma = \partial \rfloor \; dx, dx = dx_1 \wedge \ldots \wedge dx_m,$$

where $\partial = \sum_1^m e_i \partial_{x_i}$ is the Dirac operator.

There are a lot of nice, invariant differential operators of the first order between the spaces $\mathcal{E}^k$ and $\mathcal{E}^{k+1}$. The most important one is, of course, the de Rham operator d (acting componentwise). The coefficients ω_A of the form $\omega = \sum_{|A|=k} \omega_A dx_A$ being $\mathcal{C}_m$-valued functions, we can define the Dirac operator $\partial\omega$ (acting on coefficients of the form) by

$$\partial\omega = \sum_{|A|=k} (\partial\omega_A) dx_A.$$

The invariant description of $\partial\omega$ was given in [16]:

Lemma 2.1 *For any $\omega \in \mathcal{E}^k$ we have*

$$\boxed{\partial\omega = \partial\rfloor(d\omega) + d(\partial\rfloor\omega)}$$

Using the operator ∂ we get two other useful operators between $\mathcal{E}^k$ and $\mathcal{E}^{k+1}$:

$$\mathcal{D}_1\omega = dx \wedge \partial\omega, \mathcal{D}_2\omega = \partial(dx \wedge \omega). \tag{3}$$

There is a possibility to classify all Spin(m)-invariant 1-order differential operators between the spaces of forms (see [5, 2]). To explain how to do that it is better to consider spinor-valued forms (i.e. to decompose the $\mathcal{C}_m$-valued form into its spinor-valued components). So if S is the spinor space, it can be proved that (for $k \leq m/2$) the product $S \otimes \bigwedge^k R_m$ decomposes, as the Spin(m)-representation, into k irreducible parts

$$S \otimes \bigwedge^k R^m = V^{k,1} \oplus \ldots \oplus V^{k,k}. \tag{4}$$

and

$$S \otimes \bigwedge^{m-k} R^m = V^{m-k,1} \oplus \ldots \oplus V^{m-k,k}. \tag{5}$$

If we shall denote (for a fixed $k = 1, \dots, m-1$) by π_j the projection of $S \otimes \bigwedge^k R^m$ onto $V^{k,j}$ and by $\mathcal{E}^{k,j}(\Omega, S)$ the set of all maps from Ω to $V^{k,j}$, then all Spin(m)-invariant first order operators from $\mathcal{E}^k(\Omega, S)$ to $\mathcal{E}^{k+1}(\Omega, S)$ are just linear combinations of the restrictions of $\pi_j \circ d$ onto irreducible pieces $\mathcal{E}^{k,j}(\Omega, S)$.

We now want to describe the splitting of $\mathcal{E}^k(\Omega, S)$ into $\mathcal{E}^{k,j}(\Omega, S)$ using a simple Clifford notation. There are two maps $\omega \mapsto dx \wedge \omega$ and $\omega \mapsto \partial\rfloor\omega$, which could well be inverses of one another. They are not (for a general form), but we shall describe now suitable subspaces, on which the composition of them is the multiple of the identity. The main role in that will be played by the following useful identity.

Lemma 2.2 *For any $\omega \in \mathcal{E}^k(\Omega, \mathcal{C}_m)$ we have*

$$\boxed{\partial\rfloor(dx \wedge \omega) - dx \wedge (\partial\rfloor\omega) = -(n-2k).\omega}$$

The consequences are quite important. Let us denote

$$\mathcal{E}^{k,0} := \{\omega = (dx)^k f | f \in \mathcal{E}^0(\Omega, \mathcal{C}_m)\}, k = 0, \dots, m.$$

Then the maps

$$\omega \mapsto dx \wedge \omega \tag{6}$$

$$\omega \mapsto \partial\rfloor\omega \tag{7}$$

are 1-1 maps between $\mathcal{E}^{k,0}$ and $\mathcal{E}^{k+1,0}$, $k = 0, \dots, m-1$ and for all $\omega \in \mathcal{E}^{k,0}$ we have

$$\begin{aligned} dx \wedge (\partial\rfloor\omega) &= (k-m-1)k.\omega \\ \partial\rfloor(dx \wedge \omega) &= (k-m)(k+1).\omega\,. \end{aligned}$$

So the spaces $\mathcal{E}^{k,0}$ are distinguished subspaces in $\mathcal{E}^k$. For 1-forms we can find an invariant complementary subspace, defined by

$$\mathcal{E}^{1,1} := \{\omega \in \mathcal{E}^1 \,|\, \partial\rfloor\omega = 0\}.$$

It is easy to show that $\mathcal{E}^1 = \mathcal{E}^{1,0} \oplus \mathcal{E}^{1,1}$.

The space $\mathcal{E}^{1,1}$ can now be transported using the multiplications by dx to higher order forms and we shall find again the important fact that the contraction by ∂ will be (modulo a scalar multiple) inverse to the wedge product with dx.

Denote for all $k = 1, \ldots, m-1$

$$\mathcal{E}^{k,1} := \{\omega = (dx)^{k-1} \wedge \omega' \,|\, \omega' \in \mathcal{E}^{1,1}\}.$$

Then the maps

$$\begin{array}{rcl} \omega & \mapsto & dx \wedge \omega \\ \omega & \mapsto & \partial\rfloor\omega \end{array}$$

are 1-1 maps between $\mathcal{E}^{k,1}$ and $\mathcal{E}^{k+1,1}$, $k = 1, \ldots, m-2$ and we have for all $\omega \in \mathcal{E}^{k,1}$

$$\begin{array}{rcl} dx \wedge (\partial\rfloor\omega) & = & (-m+k)(k-1).\omega \\ \partial\rfloor(dx \wedge \omega) & = & (-m+k+1)k.\omega \end{array}$$

The described procedure suggests how to extend the system to get the full splitting of $\mathcal{E}^k$ for any k. Denote for all $j \leq k, j \leq m-k$

$$\mathcal{E}^{k,j} := \{\omega = (dx)^{k-j} \wedge \omega' | \omega' \in \mathcal{E}^j,\ \partial\rfloor\omega' = 0\}.$$

Then the maps

$$\begin{array}{rcl} \omega & \mapsto & dx \wedge \omega \\ \omega & \mapsto & \partial\rfloor\omega \end{array}$$

are 1-1 maps between $\mathcal{E}^{k,j}$ and $\mathcal{E}^{k+1,j}$, $k = j, \ldots, m-j-1$ and we have for all $\omega \in \mathcal{E}^{k,j}$

$$\begin{array}{rcl} dx \wedge (\partial\rfloor\omega) & = & (-m+k+j-1)(k-j).\omega \\ \partial\rfloor(dx \wedge \omega) & = & (-m+k+j)(k-j+1).\omega \end{array}$$

So we have got the splitting of $\mathcal{C}_m$-valued forms in the form ($k \leq n/2$):

$$\mathcal{E}^k = \mathcal{E}^{k,0} \oplus \ldots \oplus \mathcal{E}^{k,k}$$

$$\mathcal{E}^{m-k} = \mathcal{E}^{m-k,0} \oplus \ldots \oplus \mathcal{E}^{m-k,k}.$$

The splitting can be described by a triangle-shaped diagrams. We will show them explicitely in a few lower dimensions, the arrows will indicate maps $dx \wedge .$ and $\partial\rfloor.$. Their composition is a multiple of the identity and the factor is shown on the top of the arrow.

dim $= 3$:

$$\begin{array}{ccccccc}
\mathcal{E}^0 & \xleftrightarrow{-3} & \mathcal{E}^{1,0} & \xleftrightarrow{-4} & \mathcal{E}^{2,0} & \xleftrightarrow{-3} & \mathcal{E}^3 \\
 & & \oplus & & \oplus & & \\
 & & \mathcal{E}^{1,1} & \xleftrightarrow{-1} & \mathcal{E}^{2,1} & &
\end{array}$$

dim $= 4$:

$$\begin{array}{ccccccccc}
\mathcal{E}^0 & \xleftrightarrow{-4} & \mathcal{E}^{1,0} & \xleftrightarrow{-6} & \mathcal{E}^{2,0} & \xleftrightarrow{-6} & \mathcal{E}^{3,0} & \xleftrightarrow{-4} & \mathcal{E}^4 \\
 & & \oplus & & \oplus & & \oplus & & \\
 & & \mathcal{E}^{1,1} & \xleftrightarrow{-2} & \mathcal{E}^{2,1} & \xleftrightarrow{-2} & \mathcal{E}^{3,1} & & \\
 & & & & \oplus & & & & \\
 & & & & \mathcal{E}^{2,2} & & & &
\end{array}$$

dim $= 5$:

$$\begin{array}{ccccccccccc}
\mathcal{E}^0 & \xleftrightarrow{-5} & \mathcal{E}^{1,0} & \xleftrightarrow{-8} & \mathcal{E}^{2,0} & \xleftrightarrow{-9} & \mathcal{E}^{3,0} & \xleftrightarrow{-8} & \mathcal{E}^{4,0} & \xleftrightarrow{-5} & \mathcal{E}^5 \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \\
 & & \mathcal{E}^{1,1} & \xleftrightarrow{-3} & \mathcal{E}^{2,1} & \xleftrightarrow{-4} & \mathcal{E}^{3,1} & \xleftrightarrow{-3} & \mathcal{E}^{4,1} & & \\
 & & & & \oplus & & \oplus & & & & \\
 & & & & \mathcal{E}^{2,2} & \xleftrightarrow{-1} & \mathcal{E}^{3,2} & & & &
\end{array}$$

The spaces in horizontal rows are of the same size, they have the same dimension as $\mathcal{E}^0$ -modules. The number of pieces increases by 1 up to the middle and then decreases by 1. There is the longest column in the middle for even dimensions, while for odd dimensions there are two columns of equal length there. It can be shown using representation theory of Spin(m) that the described decomposition is just the one given by the invariant pieces in (4) and (5).

3 Monogenic differential forms.

The definition of monogenic forms will be based on the splitting of the space $\mathcal{E}^k$ of $\mathcal{C}_m$-valued k-forms on Ω into two parts:

$$\mathcal{E}^k = \mathcal{E}^{k'} \oplus \mathcal{E}^{k''}. \tag{8}$$

To motivate this definition, let us recall the definition of holomorphic forms in plane. First, there is the splitting of 1-forms

$$\mathcal{E}^1 = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$$

and $\bar{\partial}$ -operator of a function f is just the composition of the de Rham operator d and the projection onto (0,1)-part of df . We have the diagram

$$\begin{tikzcd}
 & \mathcal{E}^{0,1} \arrow[dr] & \\
\mathcal{E}^0 \arrow[ur, "\bar{\partial}"] \arrow[dr] & & \mathcal{E}^2 \\
 & \mathcal{E}^{1,0} \arrow[ur, "\bar{\partial}"] &
\end{tikzcd}$$

and holomorphic functions and holomorphic 1-forms are the kernels of the maps $\bar{\partial}$.

If the splitting (8) is chosen, we get a similar diagram

$$\begin{tikzcd}
 & \mathcal{E}^{1''} \arrow[r] \arrow[ddr] & \mathcal{E}^{2''} \cdots & \mathcal{E}^{(m-1)''} \arrow[dr] & \\
\mathcal{E}^0 \arrow[ur, "d''"] \arrow[dr] & \oplus & \oplus & \oplus & \mathcal{E}^m \\
 & \mathcal{E}^{1'} \arrow[r] \arrow[uur, "d''"] & \mathcal{E}^{2'} \cdots & \mathcal{E}^{(m-1)'} \arrow[ur, "d''"] &
\end{tikzcd}$$

where the operators d'' are defined as the composition of d with the projection onto $\mathcal{E}^{k'}_{-}$. Monogenic forms can be defined now as the kernel of the operators d''. So the question is now, if it is possible to choose the splitting (8) in such a way that the monogenic forms will have the described properties.

The requirement of invariance says us that we have to choose for the primed and double primed parts sums of invariant pieces, described in the Section 1. One such possibility was described already in [15]. It was shown there that if we take $\mathcal{E}^{k''} = \mathcal{E}^{k,0}$, then the homology of the monogenic forms describes correctly homology groups of the domain. The only disadvantage of such a choice is the fact that a suitable analogue of the Cauchy theorem is not available. To have such an analogue, it is necessary to make $\mathcal{E}^{k''}$ much bigger. But choosing them bigger, there is a danger that homology will not reproduce the topology of the domain anymore.

There is a solution for the problem. We shall choose the splitting (8) in the following way (it is necessary to discuss even and odd dimensions separately, there are some differences between them).

The case m = 2l+1.

Let us define for $1 \leq k \leq l$

$$\mathcal{E}^{k'} := \bigoplus_{\substack{0 \leq j \leq k \\ j \text{ even}}} \mathcal{E}^{k,k-j}\,,\ \mathcal{E}^{k''} := \bigoplus_{\substack{0 \leq j \leq k \\ j \text{ odd}}} \mathcal{E}^{k,k-j} \tag{9}$$

and

$$\mathcal{E}^{(m-k)'} := \bigoplus_{\substack{0 \leq j \leq k \\ j \text{ odd}}} \mathcal{E}^{m-k,k-j}\,,\ \mathcal{E}^{(m-k)''} := \bigoplus_{\substack{0 \leq j \leq k \\ j \text{ even}}} \mathcal{E}^{m-k,k-j} \tag{10}$$

Let us illustrate the definition by the example of dimension $m = 9$. All spaces involved are finite dimensional $\mathcal{E}^0$ - modules, we shall substitute in the diagram the dimension of the space $\mathcal{E}^{k,j}$ instead of the space itself. The pieces, belonging to $\mathcal{E}^{k'}$, will be indicated by boxes. In the top row the dimension of the full spaces $\mathcal{E}^k$ will be written.

dim = 9 :

$$\begin{array}{cccccccccc}
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \\
 & & & & & & & & & \\
\boxed{1} & 1 & \boxed{1} & 1 & \boxed{1} & 1 & \boxed{1} & 1 & \boxed{1} & 1 \\
 & \boxed{8} & 8 & \boxed{8} & 8 & \boxed{8} & 8 & \boxed{8} & 8 & \\
 & & \boxed{27} & 27 & \boxed{27} & 27 & \boxed{27} & 27 & & \\
 & & & \boxed{48} & 48 & \boxed{48} & 48 & & & \\
 & & & & \boxed{42} & 42 & & & &
\end{array}$$

Note that the dimensions of $\mathcal{E}^{k'}$ (for m = 9) are just binomial numbers $\binom{8}{k}$in one dimension less. In general we have

$$\dim \mathcal{E}^{k'} = \binom{m-1}{k}$$

The case m = 2l.

There are two possibilities for the definition of monogenic forms in even dimensions. The scheme is the same, but there are now two possible cases in the middle dimension, corresponding to the selfdual and anti-selfdual forms.

1. The first possibility is to define $\mathcal{E}^{k'}$ by (9) for $k \leq l$ and by (10) for $k > l$.
2. The second possibility is to define $\mathcal{E}^{k'}$ by (9) for $k < l$ and by (10) for $k \geq l$.

Note that the dimensions of the both spaces $\mathcal{E}^{l'}$ and $\mathcal{E}^{l''}$ are the same and that they equal to $\binom{m}{l}$. Note also that both possibilities differ only in the middle dimension, the spaces $\mathcal{E}^{l'}$ and $\mathcal{E}^{l''}$ are exchanged.

We shall show the both possibilities explicitely in the dimension $m = 8$.

$$
\begin{array}{ccccccccc}
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
 & & & & & & & & \\
\boxed{1} & 1 & \boxed{1} & 1 & \boxed{1} & \boxed{1} & 1 & \boxed{1} & 1 \\
 & \boxed{7} & 7 & \boxed{7} & 7 & 7 & \boxed{7} & 7 & \\
 & & \boxed{20} & 20 & \boxed{20} & \boxed{20} & 20 & & \\
 & & & \boxed{28} & 28 & 28 & & & \\
 & & & & \boxed{14} & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
 & & & & & & & & \\
\boxed{1} & 1 & \boxed{1} & 1 & 1 & \boxed{1} & 1 & \boxed{1} & 1 \\
 & \boxed{7} & 7 & \boxed{7} & \boxed{7} & 7 & \boxed{7} & 7 & \\
 & & \boxed{20} & 20 & 20 & \boxed{20} & 20 & & \\
 & & & \boxed{28} & \boxed{28} & 28 & & & \\
 & & & & 14 & & & &
\end{array}
$$

Everything is prepared now for the definition of monogenic forms.

Definition 1 *Let us consider the splitting*

$$\mathcal{E}^k = \mathcal{E}^{k'} \oplus \mathcal{E}^{k''}$$

(if $m = 2l$, we shall choose one of the two possibilities, described above). We shall denote

$$\mathcal{M}^k{}_l := \{\omega \in \mathcal{E}^{k'} \,|\, d''\omega = 0.\}$$

and we shall call such forms (left) monogenic forms.

As indicated in the definition, there is another, 'right' version of the definition. It means that the maps (6) and (7) are substituted by the right versions:

$$\begin{array}{ccc} \omega & \mapsto & \omega \wedge dx \\ \omega & \mapsto & \omega \lfloor \partial \end{array}$$

and the pieces $\mathcal{E}^{k,j}$ are defined by

$$\mathcal{E}^{k,j} := \{\omega = \omega' \wedge (dx)^{k-j} \,|\, \omega' \in \mathcal{E}^j, \omega' \lfloor \partial = 0\}.$$

The definition above can be used then in the same form, the resulting spaces will be denoted by $\mathcal{M}^k{}_r$ and such forms will be called right monogenic k-forms.

The next section will be devoted to the description of the basic properties of the monogenic forms.

4 Properties of the monogenic differential forms.

Monogenic forms were defined using invariant pieces $\mathcal{E}^{k,j}$ and invariant differential operators d''. The notion of monogenicity is hence Spin(m)-invariant. We shall state now a theorem showing that monogenic forms on a domain $\Omega \subset R^m$ can be used for the description of the topology of Ω.

Theorem 1 *Let us consider monogenic forms $\mathcal{M}^k{}_l$ given by the splitting*

$$\mathcal{E}^k = \mathcal{E}^{k'} \oplus \mathcal{E}^{k''}$$

(if $m = 2l$, we shall choose one of the two posibilities described above). Let us take any $k = 1, \ldots, m-1$.

Then d maps $\Gamma(\Omega, \mathcal{M}^k)$ into $\Gamma(\Omega, \mathcal{M}^{k+1})$ and if we denote it by d_k, we have

$$H_k(\Omega, \mathcal{C}_m) \cong Ker\, d_{k+1}/Im\, d_k.$$

A similar theorem holds, of course, also for right monogenic forms. The proof of the theorem is based on the study of operators $\mathcal{D}_1$ and $\mathcal{D}_2$, introduced in Section 2. For a given k let us denote by $d^{i,j}$ the operators $\pi_j \circ d$

restricted to $\mathcal{E}^{k,i}$ (where π_j is the projection onto $\mathcal{E}^{k+1,j}$). Then the both operators $\mathcal{D}_1$ and $\mathcal{D}_2$ can be expressed as linear combinations of $d^{i,j}$ and it is possible, using Lemma 1 and Lemma 2, to compute the coefficients explicitely. Using surjectivity of the Dirac operator, it is then possible to prove that the operator d'' maps $\mathcal{E}^k$ onto $\mathcal{E}^{k''}$. The theorem then follows from the results proved in [15].

A second important property of monogenic forms is the Cauchy theorem. In the standard version of it (see [1]) there is the product of a function and a (m-1)-form. Here we shall consider two forms in complementary dimensions and their wedge product, the theorem being that if both are monogenic (in a sense specified below), their product is closed. We shall formulate the theorem separately for even and odd dimensions, because the formulation is more complicated in the even case.

Theorem 2 *Let $m = 2l + 1$ and $k = 1, \ldots, m-2$. If $\omega \in \mathcal{M}^k{}_r$ and $\tau \in \mathcal{M}^{m-k-1}{}_l$, then*

$$d(\omega \wedge \tau) = 0.$$

To formulate the theorem in even dimensions $m = 2l$, we have to recall the notation, used in the Section 3, where $\mathcal{E}^{k'}$ were defined. There were two cases. Let us denote by $\mathcal{E}^{k'}$ the spaces defined in the first case and by $\widetilde{\mathcal{E}^{k'}}$ ones defined in the second case (it means that $\mathcal{E}^{l'}$ are defined by (9) and $\widetilde{\mathcal{E}^{l'}}$ by (10) . We shall use similar notation $\mathcal{M}^k$ and $\widetilde{\mathcal{M}^k}$ for monogenic forms in the corresponding cases. The subscripts r and l will indicate again, if we use the left or right multiplication in the definitions.

Theorem 3 *Let $m = 2l$ and $k = 1, \ldots, m-2$. If $\omega \in \mathcal{M}^k{}_r$ and $\tau \in \widetilde{\mathcal{M}^{m-k-1}}{}_l$, then*

$$d(\omega \wedge \tau) = 0.$$

The same is true also $\omega \in \widetilde{\mathcal{M}^k}{}_r$ and $\tau \in \mathcal{M}^{m-k-1}{}_l$.

The theorems above are the starting point for further development, just as in the classical case (see [1]). It opens the way to the Cauchy-Pompeju theorem, explicit indicatrices, the description of the winding numbers using monogenic forms as well as to the duality theory and general residue theory. However these questions will be treated in another paper.

References

[1] F.Brackx, R.Delanghe, F.Sommen: Clifford analysis, Research Notes in Math., 76, Pitman 1982

[2] J.Bureš, V. Souček: Generalized Cauchy-Riemann equations on Manifolds, Proc. of the 12th Winter School on Abstract Analysis 1984, Supp.-Rend.Circolo Mat.Palermo, ser.II, 6(1984), 31-42

[3] R.Delanghe: On regular analytic functions with values in Clifford algebra, Math.Ann.,185(1970),91-111

[4] R.Delanghe, F.Brackx: Duality in hypercomplex function theory, J.-Funct.Anal., 37(1978), 164-181

[5] H.D.Fegan: Conformally invariant first order differential operators, Quat.Jour.Math., 27(1976), 371-378

[6] R.Fueter: Analytische Funktionen einer Quaternionenvariablen, Comm. Math.Helv. 4(1932), 9-20

[7] K.Habetha: Eine Definition des Kroneckerindexes im R^{m+1} mit Hilfe der Cliffordanalysis, Zeitschr.Anal.Anw. 5(1986), 133-137

[8] D.Hestenes, G.Sobczyk: Clifford algebra to geometric calculus, D.Reidel, Dordrecht 1984

[9] V.Iftimie: Fonctions hypercomplexes, Bull.Math.Soc.Sc.Math. R.S. Romanie, 9(1965), 279-332

[10] P.Lounesto: Spinor function theory, Simon Stevin, 58(1984), 193-218

[11] G.C.Moisil: Les fonctions monogenes dans espaces a plusieurs dimensions, C.R.Congres des Soc.Savantes de Clermont-Ferrand, 1931

[12] J.Ryan: Clifford analysis with generalised elliptic and quasielliptic functions, Applicable Anal. 13(1982), 151-171

[13] V.Souček: H-valued differential forms on H, Proc. of the 11th Winter School, Suppl.Rend.Circolo Mat.Palermo, ser.II, 3(1984), 293-299

[14] F.Sommen: Monogenic differential forms and homology theory, Proc.-Royal Irish Ac., 84A(1984), 87-109

[15] F.Sommen, V.Souček: Hypercomplex differential forms applied to the de Rham and the Dolbeault complex, Sem.Geom. 1984, Univ. Bologna (1985), 177-192

[16] F.Sommen: Monogenic differential calculus, to appear in Trans. A.M.S.

[17] A.Sudbery: Quaternionic analysis, Proc. Camb.Phil.Soc.,85(1979),199-225

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