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DIPARTIMENTO DI MATEMATICA
SEMINARI DI GEOMETRIA 1984

REGULAR SPINOR VALUED MAPPINGS

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Introduction.

The purpose of the paper is to discuss generalizations of the standard operators ∂ and $\bar{\partial}$ in complex analysis to higher dimensional cases. There is already a lot of work done in this direction. A variety of ways was suggested for such generalizations ([1] - [10]). A general scheme, describing the operators mentioned above from the point of view of representations of groups, was presented in [11] Here we are going to discuss flat cases only (the general scheme is formulated on manifolds; a lot of results is known e.g. for Dirac a and twistor operators on manifolds [12] - [18]). The operators described here include not only elliptic operators, but also hyperbolic (or ultrahyperbolic) ones.

After the description of the group theoretical classification, we shall discuss in §.1. main examples included in the scheme. The basic facts on (complex) representations of Spin(n) are shortly summarized in §.2. and the decomposition of tensor products of such modules, needed later, are discussed in §.3. The most of examples is concerning with the case of real representation spaces and the corresponding decomposition of tensor products of real modules. This problem is treated in §.4. In §.5. we are finally able to prove that the operators, coming from the group theoretical point of view, coincide with the operators studied before in various generalizations of Cauchy-Riemann equations.

Conferenza tenuta il 3.10.1984 dal secondo Autore.

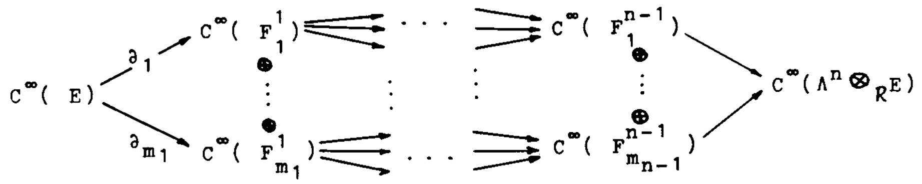
1. The classification scheme, examples.

Let us consider a real, irreducible G-module E (i.e. a representation space of G over $\mathbb{R}$). We shall concentrate to the case of the group $G = \mathrm{Spin}(p,q)$ in the paper. Let us denote by $C^\infty(V)$ the space of smooth maps from a fixed domain $\Omega \subset \mathbb{R}^n$ into the vector space V. Denoting shortly $\Lambda^j = \Lambda^j \mathbb{R}^*_n$, we can split the vector-valued de Rham sequence on Ω

$$C^\infty(\Lambda^0 \otimes_{\mathbb{R}} E) \xrightarrow{\ d\ } C^\infty(\Lambda^1 \otimes_{\mathbb{R}} E) \xrightarrow{\ d\ } \dots \xrightarrow{\ d\ } C^\infty(\Lambda^n \otimes_{\mathbb{R}} E) \qquad (1)$$

in the following way. The tensor products $\Lambda^j \otimes_{\mathbb{R}} E$ split into irreducible pieces (as G-modules over $\mathbb{R}$), say $\Lambda^j \otimes_{\mathbb{R}} E = \sum_{k=1}^{m_j} F^j_k$.

Then we have the splitted de Rham sequence



(2)

where the operators ∂_j are defined as the composition of the exterior derivative d with the projection to corresponding pieces in the decomposition. Any operator in the diagram (2) can be consider to be a generalization of ∂ or $\bar{\partial}$ (acting on functions or forms) from complex case to higher dimensions (strictly speaking it is true for Spin(n)).

Note that the same operator can appear in different diagrams for different G-modules E and that it is possible to classify them by the corresponding modules F^j_k and $F^{j+1}_{k'}$. Every such operator can appear in the first array of operators in the diagram (2) for $E = F^j_k$, so we shall discuss only operators from E to one of spaces F^1_k.

In some examples a little bit different version of the diagram (2) will be used. For complex G-modules E (i.e. representation space of G over $\mathbb{C}$) we shall consider complex valued differential forms and we shall split E-valued differential forms into complex irreducible G-modules F^j_k (i.e. $\Lambda^j_c \otimes_{\mathbb{C}} E = \sum F^j_k$).

To describe possible applications of the scheme we shall discuss now examples of various higher dimensional generalizations of C-R equations, studied by different authors. We shall present results first, i.e. we state which choice of the group G and of the G-module E leads to the corresponding differential operators. The proof of these statements will be given in §.4.

Example 1.(Regular spinor fields [5]).

Let us take $\Omega \subset R^{p,q}$, $G = \mathrm{Spin}(p,q)$ ($R^{p,q}$ has the quadratic form with p positive and q negative signs). Let E be a basic (real) G-module (see §.3.).

The tensor product $E \otimes_R \Lambda^1 (R^{p,q})^*$ splits into two pieces F_1, F_2. One of them, say F_1, is again a basic (real) spinor G-module (usually isomorphic to E). If π_1 is the projection of $\Lambda^1 \otimes_R E$ onto F_1, then the equation $(\pi_1 \circ d)\psi = 0$ is just the condition of regularity for spinor fields, described in [5]. The equation is of elliptic type for p=0 or q=0, while for other cases it is hyperbolic or ultrahyperbolic system.

Example 2 (Clifford analysis [1])

Let us take $\Omega \subset R^{0,n}$, $G = \mathrm{Spin}(n)$, $E = \mathcal{C}_n$, where $\mathcal{C}_n$ is the (real) Clifford algebra for $R^{0,n}$. The group Spin(n) is defined usually (see §.3.) as a multiplicative subgroup of $\mathcal{C}_n$, so $\mathcal{C}_n$ is the G-module (using, say, the left multiplication). The space $E = \mathcal{C}_n$ is not irreducible, however, hence in this example, strictly speaking, the described scheme cannot be used. But the space $\mathcal{C}_n$ can be decomposed into a sum of irreducible (real) G-modules (see §.3.) and all irreducible pieces in the decomposition $\mathcal{C}_n = E_1 \oplus \dots \oplus E_k$ are the same basic spinor G-modules. So we can apply the above scheme to each piece with the same result as in Example 1:

$$E_j \otimes_R \Lambda^1 = F_{j,1} \oplus F_{j,2}$$

and we shall add them together after:

$$C^\infty(E_1 \oplus \dots \oplus E_k) \cong C^\infty(\mathcal{C}_n) \begin{cases} \nearrow C^\infty(F_{1,1} \oplus .. \oplus F_{k,1}) \cong C^\infty(\mathcal{C}_n) \\ \searrow C^\infty(F_{1,2} \oplus .. \oplus F_{k,2}) \end{cases}$$

Then the operator $\pi_1 \circ d$ is just the standard differential operator for regular Clifford valued maps:

$$\psi : R^{0,n} \longrightarrow \mathcal{C}_n , \quad \psi \longmapsto \left(\sum_1^n f_i \partial/\partial x_i\right) \psi \qquad (3)$$

where $f_1, \dots, f_n$ are generators of $\mathcal{C}_n$.

It is now more common to study the following a little bit different version of regular Clifford valued maps.

Take $\Omega \subset R^{0,n+1}$, $G = Spin(n+1)$ and consider the corresponding Clifford algebra $\mathcal{C}_{n+1}$ with generators $f_0, \dots, f_n$. Then we can identify the algebra $E = \mathcal{C}_n$ with the even subalgebra $\mathcal{C}^+_{n+1}$ of $\mathcal{C}_{n+1}$ by the correspondence $e_i \sim - f_0 . f_i$; $i=1,\dots,n$.

Then the maps $\Psi: \Omega \longrightarrow \mathcal{C}_n$ are special cases of maps into $\mathcal{C}_{n+1}$ and the equation (3) takes form (after multiplication by $-f_0$)

$$(\sum_0^n e_i \partial_i) \psi = 0 , \quad \partial_i = \partial / \partial x_i$$

where $e_0 = -f_0 . f_0 = 1$. The function theory and the transform analyses for such maps is carefully studied in [1] .

The well-known Fueter's regular functions of quauaternionic variable ([9]) constitue the special case of Clifford analysis ([1]) for dimension $n=3$. In this case we have $\Omega \subset R^{0,4} = H$ and $E = \mathcal{C}_3 \cong H \oplus H$ Each of the two components of the corresponding maps are then (after a suitable identification) just Fueter regular functions.

<u>Example 3</u> (Stein-Weiss generalization of C-R equations [8])

The representation spaces E of the group $Spin(n)$ in this example are complex G-modules, maps are defined on $\Omega \subset R_n$. The most interesting examples are the cases $E = \Delta$ (where Δ is a basic (complex) spinor representation - see §.2.) and $E = \Lambda^r \otimes_R C = \Lambda^r_c$.

The procedure given by Stein and Weiss uses the fact that there is the exceptional irreducible piece in the product $E \otimes_C \Lambda^1_c$, called the Cartan product of E and Λ^1_c . It is characterized by the fact that its heighest weight is the sum of the heighest weights of E and Λ^1_c . Let us denote it by F_1 , so $E \otimes_C \Lambda^1_c = F_1 \oplus F_2$, where F_2 can be now a reducible G-module. Denoting π_2 the projection to F_2, we can write Stein and Weiss's equation as $(\pi_2 \circ d)\Psi = 0$. They contain as special cases (the Euclidean form) of Dirac equation (if E is a basic spinor representation) and the Hodge operator $d + \delta$ (if $E = \Lambda^r_c$). Both these special cases are well studied on manifolds ([13])

<u>Example 4</u> (Massless fields on Minkowski space [4])

In this example hyperbolic systems of equations will be discussed. They are not, being hyperbolic, generalization of C-R equations, but they are closely connected with the corresponding elliptic system, given by 'analytic continuation to Euclidean spacetime'. The corresponding maps are very often considered on the complexified Minkowski space ([12], [19]) Let us take $\Omega \subset R^{1,3}$ (Minkowski space), G = Lorentz group. There are two basic spinor representations Δ^+ and Δ^- of G (see §.2.) and we shall denote $E^{j,k} = S^j\Delta^+ \quad S^k\Delta^-$, where S^n denotes the n-th symmetric tensor product of the corresponding vector spaces. The decomposition of the tensor product $E^{j,k} \otimes \Lambda^1_c$ looks like

$$E^{j+1,k+1} \oplus E^{j-1,k+1} \oplus E^{j+1,k-1} \oplus E^{j-1,k-1} .$$

The four differential operators obtained in such a way are just the equations described by Garding ([4]) In the case j=0 or k=0 there are only two pieces in the decomposition.

In the standard spinor notation ([20]) the equations looks like

$$\nabla^{AA'}\Psi_{A..EA'..F'} = 0 \qquad \nabla^{A(A'}\Psi^{B'..F'}_{A\,..E} = 0$$

$$\nabla^{A'(A}\Psi^{B\,..E)}_{A'..F')} = 0 \qquad \nabla^{(A}_{(A'}\Psi^{B\,..E)}_{B'..F')} = 0$$

The special cases when the field carries only one type of indices are the most interesting ones. It is either massless field equation ([12], [19]) or twistor (or Killing) equation for spinor fields with (possibly) higher spins.([12] , [14]).

2. Tensor product of complex modules.

The classification of finite dimensional, irreducible (complex) Spin(p,q)-modules is nowadays well-known, classical fact. Tensor products of such modules decompose, as a rule, into several irreducible pieces and there are general formulae describing the decomposition ([22]). But to compute explicit examples an efficient procedure for the decomposition is needed. We have found an algorithm for it in [23] (general theorems [22] seems to be of no practical use for somebody who is not an expert, so it is very useful to have such an explicit algorithm, derived from them, at hand).

We shall review first shortly necessary facts on the heighest weight classification, then we reproduce the algorithm and use it to compute some examples needed later. We shall consider only complex modules in this paragraph.

The heighest weight classification of irreducible, finite dimensional G-modules coincide for all real forms of a complex Lie group G (see [22], chapter 8), so we shall explain it on the most convenient case of the compact Lie group G = Spin(n). Let ρ: Spin(n) $\longrightarrow$ SO(n) be the double covering of the orthogonal group. It is necessary to distinguish even and odd dimensions now.

Let k be a positive integer such that $n = 2k$ or $n = 2k+1$. Let us denote by $\tilde{T} \subset SO(n)$ the maximal torus of block diagonal matrices $(\exp(it_1),\dots,\exp(it_k))$, where $\exp(it_j)$ denotes the 2x2 real matrix

$$\begin{bmatrix} \cos t_j & -\sin t_j \\ \sin t_j & \cos t_j \end{bmatrix}$$

(for $n = 2k+1$ there is 1 in the last place on the diagonal).

Then $T = \rho^{-1}(\tilde{T})$ is the maximal torus in G. If E is a G-module, then E is a direct sum of one-dimensional T-modules. Every piece in the sum is of the form

$$(\exp(it_1),\dots,\exp(it_k)) \longrightarrow \exp i(m_1t_1+\dots+m_kt_k)$$

where components of the vector $m = (m_1,\dots,m_k) \in R^k$ are either all integral or half-integral. Such a vector is called a weight of the representation. Let us use the lexicographic ordering on the set of weights. Among all weights occuring for the given representation there is the unique highest one.

Let us denote

$n = 2k+1$: $C = \{m \mid m_1 \geq \dots \geq m_k \geq 0\}$; $C = \{m \mid m_1 > \dots > m_k > 0\}$

$n = 2k$: $C = \{m \mid m_1 \geq \dots \geq m_{k-1} \geq |m_k|\}$; $C = \{m \mid m_1 > \dots > m_{k-1} > |m_k|\}$

The weights $m \in C(C)$ are called (strictly) dominant weights.

<u>Theorem 2.1</u> ([8], [22])

There is 1-1 correspondence between irreducible, finite dimensional G-modules and dominant weights.

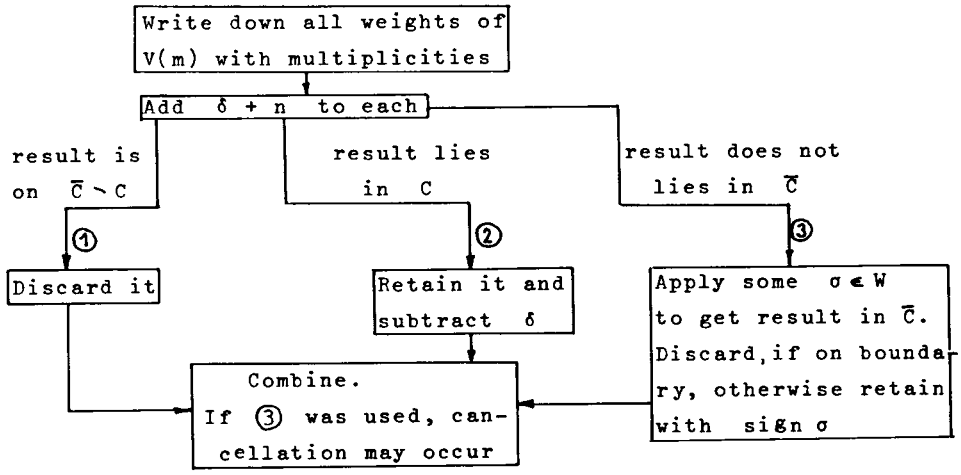
Given two G-modules E_m, E_n with heighest weights m, resp. n, then every weight of their tensor product $E_m \otimes E_n$ can be expressed in the form $\alpha + \beta$, where α (resp. β) is a weight of E_m (resp. E_n). It is very difficult, however, to find, which heighest weights will occur in the decomposition of $E_m \otimes E_n$ into irreducible pieces and what will be their multiplicity. The following algorithm for it can be found in [23].

Let us denote $\delta = (k-\frac{1}{2},k-\frac{3}{2},\dots,\frac{1}{2})$ for $n = 2k+1$

and $\delta = (k-1,k-2,\dots,0)$ for $n = 2k$.

Further we have to know all weights of the representation E_m (so it is worth to take for E_m in the algorithm the representation with more simple set of weights).

Let us denote further W the group of all permutations and sign changes acting on coordinates in R^n in the case $n = 2k+1$, while for $n = 2k$ the group W consists from all permutations together with even number of sign changes. The signum of an element W means the signum of the corresponding permutation. The diagram gives the method for computing all irreducible components $V(k_i)$ of the tensor product $V(m) \otimes V(n)$, where k_i, m, n are their heighest weights.



One of the simplest information which can be gained from the procedure is the fact that $m + n$ occurs as the heighest weight in the decomposition with the multiplicity one. The corresponding G-module E_{m+n} is called the Cartan product ([8]) or the Jung product ([22]) of E_m and E_n.

Now there is k fundamental representations of G with the property that any irreducible representation of G is obtained by succesive Cartan products from fundamental representations. The list of these fundamental modules is (we shall denote simply the modules by their heighest weights):

$n = 2k+1$:

$$E_1 = (1,0,\dots,0)$$
$$\vdots$$
$$E_{k-1} = (1,\dots,1,0)$$
$$E_k = (\tfrac{1}{2},\dots\dots,\tfrac{1}{2})$$

$n = 2k$:

$$E_1 = (1,0,\dots,0)$$
$$\vdots$$
$$E_{k-2} = (1,\dots 1,0,0)$$
$$E_{k-1} = (\tfrac{1}{2},\dots\dots,\tfrac{1}{2})$$
$$E_k = (\tfrac{1}{2},\dots,\tfrac{1}{2},-\tfrac{1}{2})$$

All these fundamental modules have the standard realizations. Modules E_j with integral components of h.w. are simply j-th exterior powers $\Lambda^j C_n$ of the basic representation of $SO(n)$ on C_n, while the other three are basic spinor modules, which are usually realized as ideals in Clifford algebras as follows.

Let $e_1,\dots,e_n$ be the standard basis in C_n. The (complex) Clifford algebra $\mathcal{C}_n^c$ is the associative algebra with unit 1 of dimension 2^n over C with generators $e_1,\dots,e_n$ satisfying relations $e_ie_j + e_je_i = 2\delta_{ij}$. There is the standard grading $\mathcal{C}_n^c = \mathcal{C}_n^+ \oplus \mathcal{C}_n^-$. They have nice realization using matrix algebras $C(k) = Mat(k\times k, C)$. For $n = 2k$ we have the isomorphism $\mathcal{C}_n^c = C(2^k)$, while for $n = 2k+1$ we have $\mathcal{C}_n^c = C(2^k) \oplus C(2^k)$.

The algebra $C(k)$ being simple, it has only one class of irreducible modules, namely representations on C_k by multiplication. This module is very often defined more invariantly to be a minimal left ideal in $C(k)$.

Because $Spin(n) \subset \mathcal{C}_n^+ \cong \mathcal{C}_{n-1}^c$, so we have two standard modules Δ^+ and Δ^- for $n = 2k$ (both of dimension 2^{k-1}), while there is only one basic module Δ (of dimension 2^k) for $n = 2k+1$.

They are just three fundamental spinor modules, described above. (For more details on Spin groups and Clifford algebras see [24]-[27])

We shall end the paragraph with a few examples of the decomposition of tensor products needed later.

<u>Example 5</u>.(see [16])

Take $E_m = C_n = (1,0,\dots,0)$ and $E_n = (\frac{1}{2},\dots,\frac{1}{2})$. Let $n = 2k$.

The set of all weights for E_m is

$(1,0,\dots,0), \dots ,(0,\dots,0,1),(-1,0,\dots,0), \dots ,(0,\dots,0,-1)$

The inspection of possibilities in the algorithm is very quick (with a little of practice) and gives the result

$$E_m \otimes E_n = (\tfrac{3}{2},\tfrac{1}{2},\dots,\tfrac{1}{2}) \oplus (\tfrac{1}{2},\dots,\tfrac{1}{2},-\tfrac{1}{2}).$$

In the same way we have

$$(1,0,\dots,0) \otimes (\tfrac{1}{2},\dots,\tfrac{1}{2},-\tfrac{1}{2}) = (\tfrac{3}{2},\tfrac{1}{2},\dots,\tfrac{1}{2},-\tfrac{1}{2}) \oplus (\tfrac{1}{2},\dots,\tfrac{1}{2})$$

Example 6 (see [16])

Take the same E_m, E_n as in Example 5. Let $n = 2k+1$.

The set of all weights for E_m now looks like

$(1,0,..,0), \ldots ,(0,..,0,1),(-1,0,..,0), \ldots ,(0,..,0,-1),(0,...,0)$.

So the algorithm gives

$(1,0,...,0)\otimes(\frac{1}{2},...,\frac{1}{2}) = (\frac{3}{2},\frac{1}{2},...,\frac{1}{2})\oplus(\frac{1}{2},...,\frac{1}{2})$.

Example 7 (see [8]).

Choose $j \in \{1,...,k\}$. Take E_m as before and

$E_{n_j} = (1,...,1,0,...,0)$ with 1 j-times.

Then the procedure gives

$E_m \otimes E_{n_j} = E_{m+n_j} \oplus E_{n_{j+1}} \oplus E_{n_{j-1}}$.

Example 8.

Take $G = Spin(3,1)$. The classification of irreducible G-modules is the same as for the group $Spin(4)$, i.e. by couples of integers or half-integers satisfying the relation $m_1 \geq |m_2|$. The fundamental modules are in this case just both spinor moduls Δ^+ and Δ^- with heighest weights $(\frac{1}{2},\frac{1}{2})$, resp. $(\frac{1}{2},-\frac{1}{2})$.

The symmetrized tensor products $S^j\Delta^+$ are succesive Cartan products of Δ^+, hence they have h.w. $(j/2,j/2)$. Similarly, the modules $S^k\Delta^-$ have h.w. $(k/2,-k/2)$. Their tensor product $E^{j,k}$ is irreducible and have hence h.w. $(\frac{1}{2}(j+k),\frac{1}{2}(j-k))$. There are simple relations $m_1 + m_2 = j$, $m_1 - m_2 = k$ between two corresponding classification of irreducible G-modules (see Example 4).

Using the algorithm it is easy to find the decomposition of the tensor product $\Lambda^1_c \otimes E^{j,k}$. There are four weights for Λ^1_c (as in Example 5) and for every weight there is the corresponding irreducible piece in the decomposition. The result is as in Example 4.

3. Real representations of Spin(p,q).

For the procedure, described in §.1., we have to consider real Spin(p,q)-modules and the decomposiotons od their tensor products over $\mathbb{R}$ with $\Lambda^1\mathbb{R}_n$. To handle the problem, we shall collect necessary notions and theorems from [21], [26], [28].

A confusion could arise as to the definition of real, resp. quaternionic G-modules. The most natural definition is the representation on a vector space over R , resp. H . But very often another definition (not completely equivalent) is used, where the module is always a complex vector space E with so called structure map j . To distinguish these two possibilities, we shall use the name G-module with real (resp. quaternionic) structure in the second case.

Definition 3.1.

Let V be a G-modul over C . A structure map on V is a map $j: V \longrightarrow V$ such that

(i) $g \circ j = j \circ g$ for all $g \in G$,

(ii) j is conjugate-linear, i.e. $j(zv) = \bar{z}(jv)$, $z \in C$, $v \in V$,

(iii) $j^2 = \pm 1$.

Definition 3.2.

(i) Let V be a G-module over R , define $cV = C \otimes_R V$, regarded as G-module over C .

(ii) Let V be a G-module over C , let rV have the same underlying set as V and the same operations from G , but regard it as a vector space over R .

(iii) Let V be a G-module over C . We define tV to have the same underlying set as V and the same operations from G , but we make C act in a new way: z acts on tV as $\bar{z}$ used to act on V.

(iv) Let V has a real structure. Then the +1 and -1 eigenspaces of j are G-modules over R , we shall denote

$$\mathrm{Re}\ V = \{v \in V \mid jv = v\} .$$

Remarks.

1. If V is a G-module over H , we may regard it as G-module over C with a structure map j , $j^2 = -1$ (actually in two ways, see [28],p. 24). Conversely, given G-module over C with a structure map j , $j^2 = -1$, we can get back the corresponding G-module over H .

2. The situation is a little bit more complicated in real case (a confusion could arise here). It is usual (see [14] , [16]) to regard G-module V' over C with a real structure j as being equivalent to a G-module V over R , if $V' = cV$ and $V = \mathrm{Re}\ V'$ (see [28] , p.25). For our purposes it will be better to distinguish clearly between them.

Theorem 3.1.([28] ,p.27)

Let V be a G-module over C , then $crV = V \oplus tV$.

Theorem 3.2 ([28] ,p.64)

Representations V with real or quaternionic structure are self-conjugate, i.e. $tV \cong V$.

Theorem 3.3 ([28] ,p.45)

If V and W are two G-modules over R such that $cV \cong cW$, then $V \cong W$.

Theorem 3.4 ([28] ,p.42)

Let $V_i, i \in I$ run over the inequivalent irreducible G-modules over R . Let m_i , n_i be nonnegative integers (all but a finite number equal to 0).

If $\bigoplus_i m_i V_i = \bigoplus_j n_j V_j$, then $m_i = n_i$, $i \in I$.

Our main interest in the paper lies in basic spinor representations of Spin(p,q) over R . To discuss them and to show their standard realization, we shall use the Clifford algebras $\mathcal{C}_{p,q}$.

Let us consider $R^{p,q}$ with the quadratic form

$$\sum_{j=1}^{p+q} \xi_j x_j^2 \, , \quad \text{where} \quad \begin{array}{l} \xi_j = +1, \ j=1,\dots,p \\ \xi_j = -1, \ j=p+1,\dots,p+q \, . \end{array}$$

The Clifford algebra $\mathcal{C}_{p,q}$ is a real associative algebra, generated by the standard basis $e_1,\dots,e_{p+q}$ of $R^{p,q}$ with relations

$$e_i e_j + e_j e_i = 2\xi_i \delta_{ij} \quad .$$

There is the Z_2-grading $\mathcal{C}_{p,q} = \mathcal{C}^+_{p,q} \oplus \mathcal{C}^-_{p,q}$.

All Clifford algebras can be realized as matrix algebras. The standard list of $\mathcal{C}_{p,q}$ ($p+q < 8$) looks like ([5],[27])

p+q \ p-q	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								R							
1							C		^2R						
2						H		$R(2)$		$R(2)$					
3					^2H		$C(2)$		$^2R(2)$		$C(2)$				
4				$H(2)$		$H(2)$		$R(4)$		$R(4)$		$H(2)$			
5			$C(4)$		$^2H(2)$		$C(4)$		$^2R(4)$		$C(4)$		$^2H(2)$		
6		$R(8)$		$H(4)$		$H(4)$		$R(8)$		$R(8)$		$H(4)$		$H(4)$	
7	$^2R(8)$		$C(8)$		$^2H(4)$		$C(8)$		$^2R(8)$		$C(8)$		$^2H(4)$		$C(8)$

Now, $\mathrm{Spin}(p,q) \subset \mathcal{C}^{+}_{p,q} \cong \mathcal{C}_{p,q-1}$, $q \geq 1$ (see [5]). The basic spinor module E for $G = \mathrm{Spin}(p,q)$ over R is any minimal left ideal in $\mathcal{C}_{p,q-1}$. The module E is moreover the vector space over C for $p-q = 2,6 \pmod 8$ and over H for $p-q = 3,4,5 \pmod 8$ (see [5]). It can be easily seen from the list of $\mathcal{C}_{p,q}$ above, any column in the corresponding matrix group being the minimal left ideal (for more invariant definition of multiplication by C ,resp. H see [5])

Let us compare now the basic G-module E with the fundamental (complex) modules $\Delta, \Delta^+, \Delta^-$. There are three different possibilities.

(i) $\mathcal{C}_{p,q-1}$ is matrix group over C (then $p+q$ is even, say $2k$). There is only one such modul E, moreover $E \cong \mathrm{Re}(\Delta^+ + \Delta^-)$, $\dim_R E = \dim_C \Delta^+ = 2^k$.

(ii) $\mathcal{C}_{p,q-1}$ is the matrix group over H . We can (by imbeeding $C \subset H$) consider the modul E as complex vector space. For $p+q=2k+1$ there is only one modul E , moreover $E \cong \Delta$ and $\dim_R E = \frac{1}{2}\dim_C \Delta = 2^{k+1}$, while for $p+q=2k$ there are two modules $E^+ \cong \Delta^+$, $E^- \cong \Delta^-$ and $\dim_R E^{\pm} = 2\dim_C \Delta^{\pm} = 2^k$.

(iii) $\mathcal{C}_{p,q-1}$ is the matrix group over R . For $p+q=2k+1$ there is only one modul E, $E \cong \mathrm{Re}\ \Delta$, $\dim_R E = \dim_C \Delta = 2^k$; while for $p+q=2k$ there are again two modules $E^+ \cong \mathrm{Re}\ \Delta^+$, $E^- = \mathrm{Re}\ \Delta^-$ and $\dim_R E = \dim_C \Delta = 2^{k-1}$.

Note that we have (using $\Delta \cong t\Delta$, $\Delta^+ \cong t\Delta^-$) in cases (i),(ii) the relation $E \otimes_R C \cong V \oplus tV$, where V is Δ, Δ^+ or Δ^- . We are able now to describe the decomposition of E-valued 1-forms.

Theorem 3.5.

Let E be a basic Spin(p,q)-module over R . Then we have the decomposition $E \otimes_R \Lambda^1(R^{p,q}) \cong F_1 \oplus F_2$, where F_1 , F_2 are summarized in the following table.

The (complex) modules $\tilde{\Delta}^+$, resp. $\tilde{\Delta}^-$, used in the table, are characterized by their heighest weights $(\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2})$, resp. $(\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})$; the modules Δ, Δ^+ , Δ^- were described in §.2.

The table of $E\otimes_R \Lambda^1(R^{p,q}) = F_1 \oplus F_2$.

field	p-q	p+q	E	F_1	F_2
R	-1,0,1 mod 8	2k+1 2k	$E\cong \mathrm{Re}\,\Delta$ $E^+\cong \mathrm{Re}\,\Delta^+$ $E^-\cong \mathrm{Re}\,\Delta^-$	$\mathrm{Re}\,\Delta$ $\mathrm{Re}\,\Delta^-$ $\mathrm{Re}\,\Delta^+$	$\mathrm{Re}\,\tilde\Delta$ $\mathrm{Re}\,\tilde\Delta^+$ $\mathrm{Re}\,\tilde\Delta^-$
H	3,4,5 mod 8	2k+1 2k	$E\cong\Delta$ $E^+\cong\Delta^+$ $E^-\cong\Delta^-$	Δ Δ^- Δ^+	$\tilde\Delta$ $\tilde\Delta^+$ $\tilde\Delta^-$
C	2,6 mod 8	2k	$E\cong \mathrm{Re}(\Delta^+\oplus\Delta^-)$	$\mathrm{Re}(\Delta^+\oplus\Delta^-)$	$\mathrm{Re}(\tilde\Delta^+\oplus\tilde\Delta^-)$

Proof.

We shall discuss separately real, complex and quaternionic cases. In each case we shall prove only one subcase (other being similar).

1. Let $E\cong \mathrm{Re}\,\Delta$. Then we have

$$c(E\otimes_R\Lambda^1(R^{p,q})^*)\cong cE\otimes_C\Lambda^1C_{p+q}\cong(\tfrac12,\dots,\tfrac12)\otimes(1,0,\dots,0)=$$
$$\cong(\tfrac12,\dots,\tfrac12)\oplus(\tfrac32,\tfrac12,\dots,\tfrac12)\cong cE\oplus c(\mathrm{Re}\,\Delta)\ .$$

We have used the fact that Λ^1C_{p+q} has real structure, hence is self-contragredient representation (see[21] ,p.141).

2. Let $E\cong\Delta$ (as complex modules). The modules $\Delta,\tilde\Delta,\Delta\oplus\tilde\Delta$ have quaternionic structure (see[21],p.143,144), it follows that $cE\cong\Delta\oplus\Delta$, $c\tilde\Delta\cong\tilde\Delta\oplus\tilde\Delta$, $c(\Delta\oplus\tilde\Delta)\cong\Delta\oplus\tilde\Delta\oplus\Delta\oplus\tilde\Delta$. So

$$c(E\otimes_R\Lambda^1(R^{p,q})^*)\cong(\Delta\oplus\Delta)\otimes_C\Lambda^1C_{p+q}\cong$$
$$\cong\left[(\tfrac12,\dots,\tfrac12)\oplus(\tfrac12,\dots,\tfrac12)\right]\otimes(1,0,\dots,0)\cong$$
$$\cong 2\Delta\oplus2\tilde\Delta=c(\Delta\oplus\tilde\Delta).$$

3. Let $E\cong \mathrm{Re}(\Delta^+\oplus\Delta^-)$, then

$$(E\otimes_R\Lambda^1(R^{p,q})^*)\cong(\Delta^+\oplus\Delta^-)\otimes_C\Lambda^1C_{p+q}\cong$$
$$\cong\left[(\tfrac12,\dots,\tfrac12)\oplus(\tfrac12,\dots,\tfrac12,-\tfrac12)\right]\otimes(1,0,\dots,0)\cong$$
$$\cong(\tfrac12,\dots,\tfrac12,-\tfrac12)\oplus(\tfrac12,\dots,\tfrac12)\oplus(\tfrac32,\tfrac12,\dots,\tfrac12)\oplus(\tfrac32,\tfrac12,.\tfrac12,-\tfrac12)$$
$$\cong\Delta^-\oplus\Delta^+\oplus\tilde\Delta^+\oplus\tilde\Delta^-=c\left[\mathrm{Re}(\Delta^+\oplus\Delta^-)\oplus\mathrm{Re}(\Delta^+\oplus\Delta^-)\right]\ .$$

The assertion of the theorem follows in all three cases from Theorem 2.3 and Theorem 2.4.

4. The coordinate description of the operators.

We are going to show now that the operators, coming from the group theoretical approach, coincide with the operators studied before in coordinate descriptions (see Examples 1 - 4). The main effort is needed in the case of regular spinor fields (Example 1).

Let us consider once more $R^{p,q}$ with the basis $e_1,\dots,e_{p+q}$ and with the form $\sum_{i=1}^{p+q} \xi_i x_i^2$, where $\xi_1 = \dots = \xi_p = 1$, $\xi_{p+1} = \dots = \xi_{p+q} = -1$. Tha basic formula, relating the spinor representation and the fundamental representation ρ on $R^{p,q}$, is

$$g\, e_i\, g^{-1} = \sum_{j=1}^{n} \rho_{ij}(g)\, e_j \quad , \quad n = p+q \tag{4}$$

Let $\varepsilon_1,\dots\varepsilon_{p+q}$ be the dual basis in $(R^{p,q})^*$, then the contragredient representation ρ^* on $\Lambda^1(R^{p,q})^*$ acts as

$$\rho^*(g) : \varepsilon_i \longrightarrow \sum_{j=1}^{n} \rho_{ji}(g^{-1})\, \varepsilon_j \quad .$$

The target space of maps, consideredin [5] is a minimal left ideal V in $\mathcal{C}_{p,q}$. The two spaces

$$E^+ = V \cap \mathcal{C}^+_{p,q} \quad , \quad E^- = V \cap \mathcal{C}^-_{p,q}$$

are both basic spinor modules of Spin(p,q), discussed in §.3. Let us denote E any of the two modules and S the spinor representation.

The representation $\rho^* \otimes S$ on $\Lambda^1 \otimes_R E$ is then described by the formula $[\rho^*(g) \otimes S(g)](\varepsilon_j \otimes v) = \sum_k \rho_{kj}(g^{-1})\varepsilon_k \otimes gv$, $g \in \mathrm{Spin}(p,q)$. We shall find now both invariant subspaces in $\Lambda^1 \otimes_R E$.

Theorem 4.1.

Denote $F = \left\{ \sum_j \varepsilon_j \otimes e_j w \;\middle|\; w \in E \right\}$ and $G = \left\{ \sum_j \varepsilon_j \otimes v_j \;\middle|\; \sum_j \xi_j e_j v_j = 0 \right\}$.

Then both F and G are invariant subspaces of $\Lambda^1 \otimes_R E$ under the action of $\rho^* \otimes S$.

Proof.

1. We have $[\rho^* \otimes S](\sum \varepsilon_j \otimes e_j w) = \sum_j \sum_k \rho_{kj}(g^{-1})\varepsilon_k \otimes g e_j v =$

$= \sum_k \varepsilon_k \otimes g[\sum_j \rho_{kj}(g^{-1}) e_j] v = \sum_k \varepsilon_k \otimes g[g^{-1} e_k g] v = \sum_k \varepsilon_k \otimes e_k g v \in F$.

2. Take $\sum_j \epsilon_j \otimes v_j \in G$, so $\sum_j \xi_j e_j v_j = 0$.

Then $[\rho^* \otimes S](\sum_j \epsilon_j \otimes v_j) = \sum_j \sum_k \rho_{kj}(g^{-1})\epsilon_k) \otimes g v_j =$

$$= \sum_k \epsilon_k \quad (\sum_j \rho_{kj}(g^{-1}) g v_j) = \sum_k \epsilon_k \otimes w_k$$

and $\sum_k \xi_k e_k w_k = \sum_k \sum_j \xi_k e_k \rho_{kj}(g^{-1}) v_j = \sum_j (\sum_k \xi_k e_k \rho_{kj}(g^{-1})) g v_j$.

Now to have the possibility to use again the relation (4) let us note that the matrices ρ in $SO(p,q)$ satisfy the relation

$$\sum_j J_{ij} \rho^{-1}{}_{jk} = \sum_j \rho_{ji} J_{jk}, \text{ where } J_{jk} = \xi_k \delta_{jk} .$$

Hence $\xi_i \rho^{-1}_{ik} = \xi_k \rho_{ki}$. So

$$\sum_k \xi_k e_k w_k = \sum_j (\sum_k \xi_j \rho_{jk}(g) e_k) g v_j = \sum_j \xi_j (g e_j g^{-1}) g v_j = g \sum_j \xi_j e_j v_j = 0 .$$

It is easy now to write projections onto both summands F and G . Let $\sum_k \epsilon_k \otimes v_k$ be an element of $\Lambda^1 \otimes_R E$. Then

$$P_F(\sum_k \epsilon_k \otimes v_k) = \sum_k \epsilon_k \otimes e_k . (\sum_j \xi_j e_j v_j) . (1/n)$$

$$P_G(\sum_k \epsilon_k \otimes v_k) = \sum_k \epsilon_k \otimes \left[v_k - e_k (\sum_j \xi_j e_j v_j)(1/n) \right]$$

Counting dimensions of F and G , it is easy to see that F has to be one of basic spinor moduls.

Going back to the Example 1, the condition regularity for spinor field $f : \Omega \longrightarrow E$ is hence

$$P_F(df) = \sum_k dx_k \otimes e_k (1/n)(\sum_j \xi_j e_j \partial f/\partial x_j) = 0 ,$$

which is (replacing some generators e_i by $-e_i$) just the condition of regularity, used in [5] .

The coordinate form of the operator arising in Example 2 follows from that in Example 1, because all irreducible pieces in the decomposition of $\mathcal{C}_n$ are isomorphic to the same spinor module and the operator has the same coordinate form for every piece.

The decompositions discussed in Examples 3 and 4 are proved in Examples 5 - 8.

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