$\mathbb{H}$ - VALUED DIFFERENTIAL FORMS ON $\mathbb{H}$

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The aim of the paper is to discuss once more the basic question of quaternionic analysis, namely what is the proper generalization of the notion of holomorphicity for functions from the field $\mathbb{H}$ of quaternions into itself. There are more possible answers to the question. The standard approach consists of attempts to generalize either the power series expansion, or Cauchy-Rieman equations or differentiability.

The first generalization poses no restriction - all real-analytic mappings from $\mathbb{H}$ to itself are allowed. The third one is too restrictive - only a subclass of linear functions satisfies the condition. The second one is the best - a suitable generalization of Cauchy-Riemann equation found and studied by Fueter ([4]) and the theory of regular functions in this sense is developed (and generalized) a lot now (for details see [2], [5], [6]).

Another approach to this basic question is presented here. The operators ∂ and $\bar\partial$ play basic role in complex analysis and they appear naturally in the discussion of complex-valued differential on $\mathbb{C}$ (or more generally on complex manifolds). It is well known, that the space $\mathcal{E}^*$ of differential forms on $\mathbb{C}$ is complexified first ($\mathcal{E}^*_c$) and the de Rham sequence $\mathcal{E}^0_c \xrightarrow{d} \mathcal{E}^1_c \xrightarrow{d} \mathcal{E}^2_c$ is splitted after into the diagram

$$\begin{tikzcd} & \mathcal{E}^{0,1}_c \arrow[dr, "\partial"] & \\ \mathcal{E}^0_c \arrow[ur, "\bar\partial"] \arrow[dr, "\partial"'] & & \mathcal{E}^2_c \\ & \mathcal{E}^{1,0}_c \arrow[ur, "\bar\partial"'] & \end{tikzcd}$$

We would like to find an analogy to this procedure in quaternionic case. In such a way we can get a new insight to the question what is the proper generalizstion of holomorphicity in quaternionic case.

1. $\mathbb{H}$ - VALUED FORMS.

Let us consider the algebra $\mathcal{E}^*$ of differential forms on $\mathbb{H}$. A typical element of $\mathbb{H}$ can be written as $q = q_0+i_1q_1+i_2q_2+i_3q_3$

(where $i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1$). Let us denote $d_j = dq_j$ and $\partial_j = \partial/\partial q_j$ $(j=0,\dots,3)$ and $q^\dagger = q_0 + i_1 q_1 + i_2 q_2 + i_3 q_3$.

The algebra $\mathcal{E}^*_{\mathbb{H}}$ of $\mathbb{H}$-valued forms on $\mathbb{H}$ is defined by $\mathcal{E}^*_{\mathbb{H}} = \mathcal{E}^* \otimes_{\mathbb{R}} \mathbb{H}$ and we shall consider it as the right vector space over $\mathbb{H}$. The exterior product is defined by the usual formula ([6]), but has unusual properties becuase of noncommutativity of coefficients. The de Rham operator d acts on such forms componentwise.

The aim is now to find a splitting of the de Rham sequence

$$\mathcal{E}^0_{\mathbb{H}} \xrightarrow{d} \mathcal{E}^1_{\mathbb{H}} \xrightarrow{d} \mathcal{E}^2_{\mathbb{H}} \xrightarrow{d} \mathcal{E}^3_{\mathbb{H}} \xrightarrow{d} \mathcal{E}^4_{\mathbb{H}}$$

i.e. to find a decomposition of $\mathcal{E}^1_{\mathbb{H}}, \mathcal{E}^2_{\mathbb{H}}, \mathcal{E}^3_{\mathbb{H}}$ in an analogy to the complex case.

2. THE DECOMPOSITION OF 1-FORMS.

To find an analogy for $dz, d\bar{z}, \partial/\partial z, \partial/\partial\bar{z}$, it is natural to consider first the forms $dq = d_0 + i_1 d_1 + i_2 d_2 + i_3 d_3$, $dq^\dagger = d_0 - i_1 d_1 - i_2 d_2 - i_3 d_3$ and operators $\partial = \frac{1}{4}(\partial_0 - i_1\partial_1 - i_2\partial_2 - i_3\partial_3)$, $\partial^\dagger = \frac{1}{4}(\partial_0 + i_1\partial_1 + i_2\partial_2 + i_3\partial_3)$. The space $\mathcal{E}^1_{\mathbb{H}}$ is 4-dimensional (as a right vector space over $\mathbb{H}$). Hence to find a suitable splitting needs to find a good basis for $\mathcal{E}^1_{\mathbb{H}}$. Two forms dq and $dq^\dagger$ are clearly not sufficient for this purpose and there is no clear candidate for missing members of a basis.

To solve the problem, let us define 1-forms $dq^1 = -i_1 dq i_1$, $dq^2 = -i_2 dq i_2$, $dq^3 = -i_3 dq i_3$. Then dq, dq^1, dq^2, dq^3 will be shown to be a basis of $\mathcal{E}^1_{\mathbb{H}}$. The field of quaternions is usually treated with fixed, chosen quaternionic units, but the basic spirit of quaternions needs an independence of such a choice (for a definition not using fixed quaternionic units see [3]). To keep such independence of the splitting we have to define

$$\mathcal{E}^{1,(1)}_{\mathbb{H}} = \{ \alpha \in \mathcal{E}^1_{\mathbb{H}} \mid \alpha = dq.f,\ f \in \mathcal{E}^0_{\mathbb{H}} \}$$

$$\mathcal{E}^{1,(3)}_{\mathbb{H}} = \{ \alpha \in \mathcal{E}^1_{\mathbb{H}} \mid \alpha = dq^1 f_1 + dq^2 f_2 + dq^3 f_3;\ f_1, f_2, f_3 \in \mathcal{E}^0_{\mathbb{H}} \}$$

Lemma 1: $\mathcal{E}^1_{\mathbb{H}} = \mathcal{E}^{1,(1)}_{\mathbb{H}} \oplus \mathcal{E}^{1,(3)}_{\mathbb{H}}$

It is easy to show that the basis d_0, d_1, d_2, d_3 can be expressed as linear combinations of dq, dq^1, dq^2, dq^3. Hence the forms dq, dq^1, dq^2, dq^3 form a basis, too.

Remark:

A new point of view is suggested by such a splitting, namely that the analogues of $\mathcal{E}^{1,0}_{\mathbb{C}}$ and $\mathcal{E}^{0,1}_{\mathbb{C}}$ in quaternionic analysis are no more similar. They have different dimensions and their similarity in complex analysis is a sort of degeneracy. As a consequence, the analogues of ∂ and $\bar{\partial}$ in quaternionic analysis will be quite different in character, too.

__Lemma 2__:

Let us denote $dq^j = -i_j dq i_j$, $\partial^j = -i_j \partial i_j$; $j = 1,2,3$.

Then $d = dq.\partial + dq^1.\partial^1 + dq^2.\partial^2 + dq^3.\partial^3$.

__Proof__: A short computation.

__Definition 1__:

Let us define operators $\mathcal{D}: \mathcal{E}^0_{\mathbb{H}} \longrightarrow \mathcal{E}^{1,(1)}_{\mathbb{H}}$, $\vec{\mathcal{D}}: \mathcal{E}^0_{\mathbb{H}} \longrightarrow \mathcal{E}^{1,(3)}_{\mathbb{H}}$ by

$\mathcal{D} f = dq.\partial f$, $\vec{\mathcal{D}} f = dq^1.\partial^1 f + dq^2.\partial^2 f + dq^3.\partial^3 f$, $f \in \mathcal{E}^0_{\mathbb{H}}$.

__Remark__.

Clearly
$$\mathcal{E}^0_{\mathbb{H}} \begin{array}{l} \xrightarrow{\mathcal{D}} \mathcal{E}^{1,(1)}_{\mathbb{H}} \\ \qquad \oplus \\ \xrightarrow{\vec{\mathcal{D}}} \mathcal{E}^{1,(3)}_{\mathbb{H}} \end{array} , \quad d = \mathcal{D} + \vec{\mathcal{D}}$$

is an analogue of the corresponding complex case picture. Let us now have a look what the solutions of $\mathcal{D} f = 0$, $\vec{\mathcal{D}} f = 0$ looks like. It is clear that $\mathcal{D} f = 0$ iff f is a (Fueter) regular function.

On the other hand we have

__Lemma 3__: $\vec{\mathcal{D}} f = 0$ iff f is (left) differentiable.

__Proof__: $\dfrac{df}{dq} = \lim\limits_{h \to 0} h^{-1}(f(q+h) - f(q))$ exists iff

iff $\partial_0 f = -i_1 \partial_1 f = -i_2 \partial_2 f = -i_3 \partial_3 f$ iff

iff $(\partial_0 f - \partial f)dx_0 = (\partial_1 f - i_1 \partial f)dx_1 = (\partial_2 f - i_2 \partial f)dx_2 = (\partial_3 f - i_3 \partial f)dx_3 = 0$

iff $df - dq.\partial f = 0$ iff $\vec{\mathcal{D}} f = 0$.

__Remark__.

Hence $\vec{\mathcal{D}} f = 0$ are just Cauchy-Riemann equations for the derivative of a quaternionic function to exist. And even if the class of solutions is very restricted, there is an interesting connection to a very nice piece of mathematics arising from some problems of mathematical physics.

Two basic differential operators are mentioned in [1] - Dirac operator and twistor operator. In the flat case (the Riemannian 4-manifold X being $\mathbb{R}_4 \simeq \mathbb{H}$) and after the identification of $\mathbb{C}_2$ with $\mathbb{H}$), we shall find that the Dirac operator can be identified with $\mathcal{D}$ and the twistor operator with $\vec{\mathcal{D}}$. (Note that differentiable quaternionic functions as well as solutions of twistor equation form a subclass of linear functions.) For details see [5].

So the presented discussion leads to the following analogy between complex and quaternionic analysis:

$\mathbb{C}$	$\mathbb{H}$
$\partial f = dz\frac{1}{2}(\partial_0 - i\partial_1)f$	$\mathcal{D}f = dq\frac{1}{4}(\partial_0 - i_1\partial_1 - i_2\partial_2 - i_3\partial_3)f$
$\bar{\partial} f = (d - \partial)f$	$\vec{\mathcal{D}}f = (d - \mathcal{D})f$
the derivative f' exists iff	the derivative f' exists iff
iff $\bar{\partial} f = 0$	iff $\vec{\mathcal{D}} f = 0$
then $f' = \frac{1}{2}(\partial_0 - i\partial_1)f$	then $f' = \frac{1}{4}(\partial_0 - i_1\partial_1 - i_2\partial_2 - i_3\partial_3)f$

3. THE DECOMPOSITION OF 2-FORMS.

The natural pocedure for the decomposition of 2-forms could be to split again the de Rham operator $d = \mathcal{D} + \vec{\mathcal{D}}$ and to use the multiplicative properties of $\mathbb{H}$-valued forms to find a natural decomposition of 2-forms. Such a try would end in a complete maes. But there is another possibility - let us decompose the de Rham operator d in another way:

$$d = \mathcal{D}^{\dagger} + \vec{\mathcal{D}}^{\dagger}, \text{ where } \mathcal{D}^{\dagger} = dq^{\dagger}.\partial^{\dagger},\ \vec{\mathcal{D}}^{\dagger} = dq^{1\dagger}\partial^{1\dagger} + dq^{2\dagger}\partial^{2\dagger} + dq^{3\dagger}\partial^{3\dagger}.$$

(There is another reason, connected with the transformation properties with respect to orthogonal transformation, suggesting that it is a good choice.) The multiplicative properties of considered 1-forms leads us then to the following picture:

Let us denote
$$\omega_1 = d_0\wedge d_1 + d_2\wedge d_3, \quad \bar{\omega}_1 = d_0\wedge d_1 - d_2\wedge d_3$$
$$\omega_2 = d_0\wedge d_2 + d_3\wedge d_1, \quad \bar{\omega}_2 = d_0\wedge d_2 - d_3\wedge d_1$$
$$\omega_3 = d_0\wedge d_3 + d_1\wedge d_2, \quad \bar{\omega}_3 = d_0\wedge d_3 - d_1\wedge d_2$$

Then
$$dq\wedge dq^{\dagger} = -(i_1\omega_1 + i_2\omega_2 + i_3\omega_3)$$
$$dq\wedge dq^{1\dagger} = -i_1\bar{\omega}_1$$
$$dq^{1}\wedge dq^{\dagger} = -i_1\bar{\omega}_1$$
$$dq^{1}\wedge dq^{1\dagger} = -i_1\omega_1 + i_2\omega_2 + i_3\omega_3 \qquad \text{and cyclic permutation}$$
$$dq^{1}\wedge dq^{2\dagger} = i_3\bar{\omega}_3 \qquad \text{of them}$$
$$dq^{2}\wedge dq^{1\dagger} = i_3\bar{\omega}_3$$

It suggest the following definition:

<u>Definition 2</u>:

Let us denote
$$\mathcal{E}^{2,(+)}_{\mathbb{H}} = \{i_1\omega_1 f_1 + i_2\omega_2 f_2 + i_3\omega_3 f_3 \mid f_1, f_2, f_3 \in \mathcal{E}^{\bullet}_{\mathbb{H}}\}$$
$$\mathcal{E}^{2,(-)}_{\mathbb{H}} = \{i_1\bar{\omega}_1 f_1 + i_2\bar{\omega}_2 f_2 + i_3\bar{\omega}_3 f_3 \mid f_1, f_2, f_3 \in \mathcal{E}^{\bullet}_{\mathbb{H}}\}$$

and
$$\mathcal{D}^{\dagger}(dq.f) = dq\wedge\mathcal{D}^{\dagger}f = -(i_1\omega_1 + i_2\omega_2 + i_3\omega_3)\partial^{\dagger}f$$
$$\vec{\mathcal{D}}^{\dagger}(dq.f) = dq\wedge\vec{\mathcal{D}}^{\dagger}f = -i_1\bar{\omega}_1\partial^{1}f - i_2\bar{\omega}_2\partial^{2}f - i_3\bar{\omega}_3\partial^{3}f$$

while
$$\sigma(dq^1 f_1 + dq^2 f_2 + dq^3 f_3) = i_1\bar{\omega}_1(-\partial^{\dagger}f_1 + \partial^{3\dagger}f_2 + \partial^{2\dagger}f_3) + $$
$$+ i_2\bar{\omega}_2(-\partial^{\dagger}f_3 + \partial^{1\dagger}f_2 + \partial^{2\dagger}f_1) + i_3\bar{\omega}_3(-\partial^{\dagger}f_3 + \partial^{1\dagger}f_2 + \partial^{2\dagger}f_1),$$

$$\sigma'(dq^1f_1+dq^2f_2+dq^3f_3)= i_1\omega_1(-\partial^{1\dagger}f_1+\partial^{2\dagger}f_2+\partial^{3\dagger}f_3) + $$
$$+i_2\omega_2(\partial^{1\dagger}f_1-\partial^{2\dagger}f_2+\partial^{3\dagger}f_3) + i_3\omega_3(\partial^{1\dagger}f_1+\partial^{2\dagger}f_2-\partial^{3\dagger}f_3)$$

where $d = \sigma + \sigma'$ is a splitting of d on $\mathcal{E}^{1,(3)}_{\mathbb{H}}$.

<u>Remarks</u>.

1. Altogether we have

$$\begin{tikzcd}
\mathcal{E}^{1,(1)}_{\mathbb{H}} \arrow[r, "\mathcal{D}^\dagger"] \arrow[dr, "\bar{\mathcal{D}}^\dagger"] & \mathcal{E}^{2,(+)}_{\mathbb{H}} \\
\mathcal{E}^{1,(3)}_{\mathbb{H}} \arrow[ur, "\sigma'"] \arrow[r, "\sigma"'] & \mathcal{E}^{2,(-)}_{\mathbb{H}}
\end{tikzcd}$$

2. The spaces $\mathcal{E}^{2,(+)}_{\mathbb{H}}$ and $\mathcal{E}^{2,(-)}_{\mathbb{H}}$ are just the spaces of ($\mathbb{H}$-valued) selfdual and antiselfdual 2-forms. It brings us once more to the mentioned paper [1], where the spaces of selfdual connections on special spaces are studied.

3. The image of the operator $\mathcal{D}^\dagger$ lies in a distinguished subspace of $\mathcal{E}^{2,(+)}$ defined by $\{(i_1\omega_1+i_2\omega_2+i_3\omega_3).f \mid f \in \mathcal{E}^{0}_{\mathbb{H}}\}$.

4. THE DECOMPOSITION OF 3-FORMS.

There is a very useful $\mathbb{H}$-valued 3-form (used e.g. in the Cauchy integral formula - see [5])

$$Dq = d_1\wedge d_2\wedge d_3 - i_1 d_0\wedge d_2\wedge d_3 - i_2 d_0\wedge d_3\wedge d_1 - i_3 d_0\wedge d_1\wedge d_2$$

and we shall define further $Dq^j = -i_j Dq\, i_j$, $j=1,\dots,3$.

After the splitting $d = \mathcal{D}+\bar{\mathcal{D}} : \mathcal{E}^2_{\mathbb{H}} \longrightarrow \mathcal{E}^3_{\mathbb{H}}$ the following multiplicative properties will be found:

$i_1\omega_1\wedge dq = Dq$

$i_1\bar{\omega}_1\wedge dq = Dq^1$

$i_1\omega_1\wedge dq^1=-Dq^1$ and other (cyclic) permutations

$i_1\bar{\omega}_1\wedge dq^1=-Dq$

$i_1\omega_1\wedge dq^2=i_3\omega_3\wedge dq^2=Dq^2$

$i_1\bar{\omega}_1\wedge dq^2=i_2\bar{\omega}_2\wedge dq^1=-Dq^3$

This leads to the following definition:

<u>Definition 3</u>:

Let us define
$$\mathcal{E}^{3,(1)}_{\mathbb{H}} = \{Dq.f \mid f \in \mathcal{E}^{0}_{\mathbb{H}}\}$$
$$\mathcal{E}^{3,(3)}_{\mathbb{H}} = \{Dq^1f_1+Dq^2f_2+Dq^3f_3 \mid f_1,f_2,f_3\in\mathcal{E}^{0}_{\mathbb{H}}\}$$

It is easy to show that $\mathcal{E}^3_{\mathbb{H}} = \mathcal{E}^{3,(1)}_{\mathbb{H}} \oplus \mathcal{E}^{3,(3)}_{\mathbb{H}}$.

We shall define $\mathcal{D}(\sum_1^3 i_j\omega_j f_j) = Dq.\partial(f_1+f_2+f_3)$,

$$\bar{\mathcal{D}}(\sum_1^3 i_j\omega_j f_j) = Dq^1\partial^1(-f_1+f_2+f_3) + Dq^2\partial^2(f_1-f_2+f_3) + Dq^3\partial^3(f_1+f_2-f_3),$$

on $\mathcal{E}^{2,(+)}_{\mathbb{H}}$ and

$$\tau(\sum_1^3 i_j\bar{\omega}_j f_j) = Dq^1(\partial f_1-\partial^3 f_2-\partial^2 f_3)+Dq^2(\partial f_2-\partial^1 f_3-\partial^3 f_1)+$$
$$+ Dq^3(\partial f_3-\partial^1 f_2-\partial^2 f_1),$$

$\tau'(\sum_1^3 i_j\bar{\omega}_j f_j) = -\,Dq(\partial^1 f_1+\partial^2 f_2+\partial^3 f_3)$ on $\mathcal{E}^{2,(-)}_{\mathbb{H}}$.

Hence we have the picture:

$$\begin{tikzcd}
\mathcal{E}^{2,(+)}_{\mathbb{H}} \arrow[r, "\mathcal{D}"] \arrow[ddr, "\overrightarrow{\mathcal{D}}" near start] & \mathcal{E}^{3,(1)}_{\mathbb{H}} \\
\oplus & \oplus \\
\mathcal{E}^{2,(-)}_{\mathbb{H}} \arrow[uur, "\tau'" near start] \arrow[r, "\tau"'] & \mathcal{E}^{3,(3)}_{\mathbb{H}}
\end{tikzcd}.$$

To complete the picture we shall use the decomposition $d=\mathcal{D}^\dagger+\overrightarrow{\mathcal{D}}^\dagger$ on 3-forms. The multiplicative properties of $\mathbb{H}$-valued forms give us the following lemma.

<u>Lemma 4</u>:

The operator $\mathcal{D}^\dagger$ equals to zero on $\mathcal{E}^{3,(3)}_{\mathbb{H}}$, while $\overrightarrow{\mathcal{D}}^\dagger$ is zero on $\mathcal{E}^{3,(1)}_{\mathbb{H}}$.
Moreover $\overrightarrow{\mathcal{D}}^\dagger(\sum_1^3 Dq^j f_j) = -4d^4q(\partial^{1\dagger}f_1+\partial^{2\dagger}f_2+\partial^{3\dagger}f_3)$,
while $\mathcal{D}^\dagger(Dq.f) = -4d^4q\,\partial^\dagger f$,
where $d^4q = d_0\wedge d_1\wedge d_2\wedge d_3$.

<u>Proof</u>:

A short computation gives
$Dq\wedge dq^\dagger = -4d^4q$, $Dq\wedge dq^{1\dagger} = 0$, $Dq^1\wedge dq^\dagger = Dq^1\wedge dq^{2\dagger} = 0$, $Dq^1\wedge dq^{1\dagger} = -4d^4q$.

The whole scheme looks like follows:

$$\begin{tikzcd}
 & \mathcal{E}^{1,(1)}_{\mathbb{H}} \arrow[r, "\mathcal{D}^\dagger"] \arrow[dr, "\overrightarrow{\mathcal{D}}^\dagger"' near start] & \mathcal{E}^{2,(+)}_{\mathbb{H}} \arrow[r, "\mathcal{D}"] \arrow[dr, "\overrightarrow{\mathcal{D}}"' near start] & \mathcal{E}^{3,(1)}_{\mathbb{H}} \arrow[dr, "\mathcal{D}^\dagger"] & \\
\mathcal{E}^{c}_{\mathbb{H}} \arrow[ur, "\mathcal{D}"] \arrow[r, "\overrightarrow{\mathcal{D}}"'] & \mathcal{E}^{1,(3)}_{\mathbb{H}} \arrow[ur, "\sigma'" near start] \arrow[r, "\sigma"'] & \mathcal{E}^{2,(-)}_{\mathbb{H}} \arrow[ur, "\tau'" near start] \arrow[r, "\tau"'] & \mathcal{E}^{3,(3)}_{\mathbb{H}} \arrow[r, "\overrightarrow{\mathcal{D}}^\dagger"] & \mathcal{E}^{4}_{\mathbb{H}}
\end{tikzcd}$$

which is claimed to be the proper analogy of the complex case picture.

<u>Remarks</u>.

1. All scheme can be extended to a more general setting, e.g it would much more interesting to study such questions on manifolds. These questions will be studied in another paper.
2. Some information on the topology of domains in $\mathbb{C}$ can be gained using the square of ∂ and $\bar{\partial}$ operators in complex case. Similar information in quaternionic case is to be found yet.

REFERENCES

[1]
ATIYAH M.F., HITCHIN N.J., SINGER I.M. "Self-duality in four-dimensional Riemannian geometry", Proc. Roy. Soc. London Ser. A 362(1978), 425-461

[2]
DELANGHE R., BRACKX F., SOMMEN F. "Clifford analysis", Research Notes in Math., Pitman, 1982

[3] EHLERS J., RINDLER W., ROBINSON I. "Quaternions, Bivectors and the Lorentz group", in Perspectives in Geometry and Relativity, vol. in honour of Prof. Hlavatý, 1966

[4] FUETER R. "Die Funktionentheorie der Differentialgleichungen $\Delta u=0$ und $\Delta\Delta u=0$ mit Vier Reallen Variablen", Com. Math. Helvetici 7, (1934-35), 307-330

[5] SOUČEK V. "Holomorphicity in quaternionic analysis", to be published in Seminario di variabili complesse, Bologna, 1982

[6] SUDBERY A. "Quaternionic analysis", Math. Proc. of the Camb. Phil. Soc. 85(1979), 199-225

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