(1)

(Aspects of) Riemann surfaces: (include)

19-th century math

- complex function theory (in 1 complex variable)
- geometric analysis (PDEs on manifolds)
- algebraic geometry (function fields of deg tr = 1)
- diff. geometry (surfaces theory)
- algebraic topology of surfaces (genus, Euler characteristic)
- mathematical physics (integrable systems, string theory)

The concept of Riemann surface originates in the study of alg. fions and their integrals. An easy example is the pendulum equation:

$$\alpha = \alpha(t),\ t \in \mathbb{R}_+ \quad : \quad \frac{d^2\alpha}{dt^2} = -\sin\alpha .$$

(the length of pendulum of acceleration due to gravity $\rightsquigarrow$ absorbed in the redefinition of t.)

Assuming $E := \frac{1}{2}\left(\frac{d\alpha}{dt}\right)^2 - \cos\alpha$ is fixed energy (i.e., E is t-independent, or, constant of motion) want to solve $\alpha = \alpha(t)$. We have 1st order ODE

$$\frac{d\alpha}{dt} = \sqrt{2(E+\cos\alpha)}$$

$\Rightarrow$ (separation of variables) $$t = \int \frac{d\alpha}{\sqrt{2(E+\cos\alpha)}} .$$

Now, we have to

- solve (= compute) the previous integral to get $t = t(\alpha)$,
- invert the function $\alpha \mapsto t(\alpha)$ to produce a function $t \to \alpha(t)$.

The substitution $\alpha \mapsto u(\alpha) := \sin\frac{\alpha}{2}$ transforms the last integral into

$$t = \int \frac{du}{\sqrt{(1-u^2)\left(\frac{E+1}{2} - u^2\right)}} = \int \frac{du}{\sqrt{(1-u^2)(k^2-u^2)}} \qquad k = \frac{E+1}{2}$$

$$\overset{u=kx}{=} \int \frac{dx}{\sqrt{(1-k^2x^2)(1-x^2)}}$$

This integral can't be solved in terms of elementary functions. In fact, the Jacobi elliptic sn-function is defined by

$$x = \mathrm{sn}(t,k) \quad : \quad t = \int_0^x \frac{d\xi}{\sqrt{(1-k^2\xi^2)(1-\xi^2)}} \quad)$$

and this is the solution of pendulum equation.

What this has to do with Riemann surfaces?

(2) Firstly, look at some functions (algebraic) we can/can not integrate:

1/ $\int p(x)\,dx$ for $p(x) = a_0 + a_1 x + \dots + a_n x^n$ a polynomial is easy,

2/ $\int R(x)\,dx$, $R(x) = \frac{p(x)}{q(x)} \in \mathbb{C}(x)$, is solved by partial fraction decomposition,

3/ $\int R(x, \sqrt{x-a})\,dx$, $R(x,y)$ is a rational fcion of x and y, can be reduced to 2/ by substitution $y = \sqrt{x-a}$

4/ $\int R\left(x, \sqrt{(x-a)(x-b)}\right)dx$, $a \neq b$, can be reduced to 2/ by substitution $y = \sqrt{\frac{x-a}{x-b}}$

5/ $\int R\left(x, \sqrt{(x-a)(x-b)(x-c)}\right)dx$, $a \neq b \neq c$, can't be solved in elementary functions

6/ $\int R(x, \sqrt{p(x)})\,dx$, p a polynomial with different roots and $\deg(p) > 2$, can't be solved in elementary functions.

So $\int R(x, \sqrt{p(x)})\,dx$ for $\deg p \leq 2$ can / $\deg p > 2$ can't be solved in elementary fcions - is there a deeper reason for this?

Consider the function $\sqrt{p(x)}$ for complex variable x.

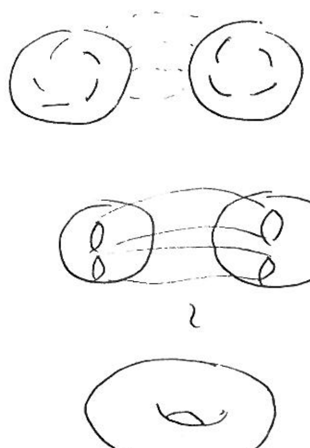
The expression $\sqrt{x-a}$ has two values $\forall x \in \mathbb{C}$, except $a \in \mathbb{C}$. We can't choose one of the values consistently on $\mathbb{C}$ to obtain a holomorphic fcion. But if we cut $\mathbb{C}$ along the line from a to "∞", we can choose one of the values consistently and obtain holomorphic fcion on $\mathbb{C} \setminus \{\langle a, \infty)\}$. Take another copy of $\mathbb{C} \setminus \{\langle a, \infty)\}$, and consider on it the opposite value of square root of $x-a$. Now glue together the two copies of $\mathbb{C} \setminus \{\langle a, \infty)\}$ along the cuts where the two functions $\sqrt{x-a}$ and $-\sqrt{x-a}$ take the same value. The result is the Riemann surface of $\sqrt{x-a}$, and topologically it is a sphere.

For $\sqrt{(x-a)(x-b)}$, proceed analogously, cut by the line $\langle a, b \rangle$
(Again, the Riemann surface is topologically the sphere.)

To define $\sqrt{(x-a)(x-b)(x-c)\cancel{(x-d)}}$ as a holomorphic function, we need to cut (e.g.) from a to b and c to ∞. If we take the two spheres and glue where the values coincide, we get the surface which is topologically torus.

The case of $\sqrt{(x-a)(x-b)(x-c)(x-d)}$ is similar as the previous case, except we cut from a to b and c to d.

In general, the Riemann surface for $\sqrt{p(x)}$, $\deg(p) = 2n$ or $2n-1$ with different roots, is obtained by gluing two spheres with n cuts. This is topologically a surface of genus $n-1$



RS - Basic definitions

X ... a topological space

Def 1 (complex charts) A complex chart on X is a homeomorphism

$\varphi : U \to V$, $\quad U \subseteq X$, $V \subseteq \mathbb{C}$ open sets. U .. domain of the chart φ.

φ is centered at $p \in U$ if $\varphi(p) = 0 \in \mathbb{C}$.

Think of φ as giving a (local) complex coordinate on U:
$\varphi(x) = z$ for $x \in U$, $z \in \mathbb{C}$. Allows (local) function calculus in the coordinate z.

(4) Ex: $X = \mathbb{R}^2$, $U \subseteq \mathbb{R}^2$ arbitrary $\varphi_U(x,y) = \dfrac{x}{1+\sqrt{x^2+y^2}} + i \dfrac{y}{1+\sqrt{x^2+y^2}}$ are complex charts on $\mathbb{R}^2$.

Ex: $\left.\begin{array}{l} \varphi : U \to V \text{ a complex chart on } U \\ \psi : V \to W \text{ a holomorphic bijection} \\ \quad\; \cap \quad\;\; \cap \\ \quad\; \mathbb{C} \quad\;\; \mathbb{C} \end{array}\right\}$ $\psi \circ \varphi : U \to W$ a complex chart on U

("change of coordinates")

Def 2: (difference between charts)

$\left.\begin{array}{l} \varphi_1 : U_1 \to V_1 \\ \varphi_2 : U_2 \to V_2 \end{array}\right\}$ complex charts on X

φ_1, φ_2 are compatible if either $U_1 \cap U_2 = \emptyset$ or

$$\begin{array}{ccc} \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) & \to & \varphi_2(U_1 \cap U_2) \\ \cap & & \cap \\ V_1 & \dashrightarrow & V_2 \\ \cap & & \cap \\ \mathbb{C} & & \mathbb{C} \end{array}$$

is holomorphic.

The definition is symmetric: if $\varphi_2 \circ \varphi_1^{-1}$ is holom. on $\varphi_1(U_1 \cap U_2)$, then $\varphi_1 \circ \varphi_2^{-1}$ is holomorphic on $\varphi_2(U_1 \cap U_2)$. The function $\varphi_2 \circ \varphi_1^{-1}$ is transition function for $(U_1, \varphi_1), (U_2, \varphi_2)$. It is a bijection.

Lemma 3 (Property of transition fcion)

Let T be a transition map between two compatible ~~holom.~~ charts.

Then the derivative T' is never zero in the domain of T.

$\mathbb{C} \to \mathbb{C}$

Pf: S ... inverse of T on $\mathrm{dom}(T)$, i.e. $S \circ T = \mathrm{Id}_{\mathrm{dom}(T)}$.

This means $S(T(w)) = w \;\forall\, w \in \mathrm{dom}(T)$. The derivative w.r. to w: $S'(T(w))\, T'(w) = 1 \Rightarrow$ $T'(w) \neq 0 \;\; \forall\, w \in \mathrm{dom}(T)$. ▨

T .. transition fcion between charts φ, ψ, p a point in their common domain. Denote $\left.\begin{array}{l} z = \varphi(x) \\ w = \psi(x) \end{array}\right\}$ local coordinates, $\begin{array}{l} z_0 = \varphi(p) \\ w_0 = \psi(p) \end{array}$

The lemma 3 then tells us

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$T = \varphi \circ \psi^{-1}$ is of the form

$$z = T(w) = z_0 + \sum_{n \geq 1} a_n (w - w_0)^n, \quad \text{with } a_1 \neq 0$$

Example: $S^2 \subseteq \mathbb{R}^3$ the unit 2-sphere in $\mathbb{R}^3$,

$$S^2 = \{(x, y, w) \in \mathbb{R}^3 \mid x^2 + y^2 + w^2 = 1\}.$$

The $w = 0$ hyperplane is isomorphic to $\mathbb{C}$: $(x, y, 0) \to x + iy$

let $\varphi_1 : S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$ be the stereographic projection from $(0,0,1)$,

$$\varphi_1(x, y, w) = \frac{x}{1-w} + i\frac{y}{1-w}.$$

The inverse to φ_1 is

$$\varphi_1^{-1}(z) = \left(\frac{2\operatorname{Re}(z)}{|z|^2+1}, \frac{2\operatorname{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right).$$

Define

$\varphi_2 : S^2 \setminus \{(0,0,-1)\} \to \mathbb{C}$ by projection from $(0,0,-1)$ followed by the complex conjugation:

$$\varphi_2(x, y, w) = \frac{x}{1+w} - i\frac{y}{1+w},$$

with the inverse

$$\varphi_2^{-1}(z) = \left(\frac{2\operatorname{Re}(z)}{|z|^2+1}, \frac{-2\operatorname{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1} \right).$$

The common domain is $S^2 \setminus \{(0,0,1), (0,0,-1)\}$, and is mapped by both φ_1, φ_2 to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The composition $(\varphi_2 \circ \varphi_1^{-1})(z) = \frac{1}{z}$, which is holomorphic. So the charts are compatible.

<u>Def 4</u>: A complex atlas $\mathcal{A}$ on X is a collection $\mathcal{A} := \{\underbrace{\varphi_\alpha : U_\alpha \to V_\alpha}_{\text{pair-wise compatible complex charts}}\}$, such that $X = \bigcup_\alpha U_\alpha$

Two complex atlases $\mathcal{A}, \mathcal{B}$ are equivalent if every chart of one atlas is compatible with every chart of the other. Another way to say that - two atlases are equivalent iff their union is a complex atlas. A Zorn's lemma then implies that $\forall$ atlas is contained in a unique maximal complex atlas.

<u>Def 5</u>: A complex structure on X is a maximal complex atlas on X, or equivalently, an equivalence class of complex atlases on X.

<u>Def 6</u>: (Riemann surface) A Riemann surface is a second countable connected Hausdorff topological space together with a complex structure.

<u>Remark</u>: As it follows from the definition of 2-dim real manifold with the transition maps (C^∞ - functions $\mathbb{R}^2 \to \mathbb{R}^2$), Riemann surfaces are orientable 2-dim real differentiable manifolds.

<u>Examples of Riemann surfaces</u>

We know that, given an open cover $\{U_\alpha\}_\alpha$ of a top. space X, a subset $U \subseteq X$ is open in X iff $U \cap U_\alpha$ is open in U_α $\forall \alpha$. Then $U_\alpha \cap U_\beta$ is open in U_α $\forall \alpha, \beta$, and because φ_α is open $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α $\forall \alpha, \beta$. ($V_\alpha = \varphi_\alpha(U_\alpha)$.)

There is the following route to define Riemann surfaces:

- consider a set X, $\{U_\alpha\}_\alpha$ a countable cover of X,

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- $\forall \alpha$ find a bijection $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{C}$;
- check $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α $\forall \alpha, \beta$. ($\Rightarrow$ a topology on X s.t. $\forall$ U_α is open and φ_α is a complex chart on X.)
- check $\{\varphi_\alpha\}_\alpha$ are pairwise compatible;
- — " — X is connected and Hausdorff;

A/ The <u>projective line</u> $\mathbb{CP}^1 \overset{\text{homeom}}{\simeq} S^2 \overset{\text{homeom}}{\simeq} \mathbb{C} \cup \{\infty\}$ is the set of 1-dim (complex) subspaces of $\mathbb{C}^2$. If $(z, w) \in \mathbb{C}^2$ is non-zero vector in $\mathbb{C}^2$, its span is a point in $\mathbb{CP}^1$. We denote the span of (z, w) by $[z : w]$, and $[\lambda z : \lambda w] = [z : w]$ $\forall \lambda \in \mathbb{C}^*$.

The complex structure on $\mathbb{CP}^1$: introduce

$$U_0 := \{[z : w] \mid z \neq 0\}, \quad U_1 := \{[z : w] \mid w \neq 0\},$$

so U_0, U_1 cover $\mathbb{CP}^1$. Define

$$\left.\begin{aligned} &\varphi_0 : U_0 \to \mathbb{C}, \quad \varphi_0([z : w]) = \frac{w}{z}, \\ &\varphi_1 : U_1 \to \mathbb{C}, \quad \varphi_1([z : w]) = \frac{z}{w}; \end{aligned}\right\} \varphi_0, \varphi_1 \text{ are bijections}$$

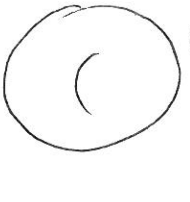
Note $\varphi_i(U_0 \cap U_1) = \mathbb{C}^*$, $i = 1, 2$, open in $\mathbb{C}$. The transition fcns

$\varphi_1 \circ \varphi_0^{-1}$ maps $s \to \frac{1}{s}$, the holomorphic fcn $\mathbb{C}^* \to \mathbb{C}^*$

hence (U_0, φ_0), (U_1, φ_1) are compatible.

U_0, U_1 connected, $U_0 \cap U_1 \neq \emptyset$ $\Rightarrow$ $U_0 \cup U_1$ connected as well.

As for Hausdorff property: if $p, q \in U_0$ or $p, q \in U_1$, they are separated by open sets since U_i are Hausdorff. Similarly for $p \in U_0 \setminus U_1$ and $q \in U_1 \setminus U_0$, which are separated by $\varphi_0^{-1}(D)$ and $\varphi_1^{-1}(D)$ (D is an open unit disc.) $\mathbb{CP}^1$ is the union of the two closed sets $\varphi_0^{-1}(\bar{D})$, $\varphi_1^{-1}(\bar{D})$, $\bar{D}$ closed unit disc in $\mathbb{C}$. Since $\bar{D}$ is compact, $\mathbb{CP}^1$ is compact as well.



$\mathbb{CP}^1 \simeq S^2$

B/ The complex tori : $\omega_1, \omega_2 \in \mathbb{C}^*$, $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. Define L to be a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{ m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z} \} \subseteq \mathbb{C}$ abelian group.

$X := \mathbb{C}/L$, $\pi : \mathbb{C} \to X$ projection. X has the quotient topology ($U \subseteq X$ is open iff $\pi^{-1}(U) \subseteq \mathbb{C}$ is open). π is continuous, and $\mathbb{C}$ is connected $\Rightarrow$ X is connected.

$\forall$ open set $U \subseteq X$ is the image of an open set in $\mathbb{C}$, namely $\pi^{-1}(U)$: $\pi(\pi^{-1}(U)) = U$. π is open mapping - if $V \subseteq \mathbb{C}$ is open in $\mathbb{C}$, to check $\pi(V)$ is open in X is equivalent to show $\pi^{-1}(\pi(V))$ is open in $\mathbb{C}$: we have $\pi^{-1}(\pi(V)) = \bigcup_{\omega \in L} (V + \omega)$,

the union of translates of V, obviously an open subset of $\mathbb{C}$.

$\forall z \in \mathbb{C}$ define the parallelogram $P_z := \{ z + \lambda_1\omega_1 + \lambda_2\omega_2 \mid \lambda_i \in \langle 0,1 \rangle \}$.

Any point of $\mathbb{C}$ is congruent mod L to a point of P_z, so π maps P_z onto X. Since P_z is compact, so is X.

$L \subseteq \mathbb{C}$ is a discrete subset, i.e., $\exists \epsilon > 0$ s.t. $|\omega| > 2\epsilon$ $\forall$ $\omega \in L^*$ (non-zero elements of L.) For $\epsilon > 0$, $z_0 \in \mathbb{C}$, $D = D(z_0, \epsilon)$ is the open disc at z_0 of radius ϵ. In particular, no two points of $D(z_0, \epsilon)$ differ by an element of L.

We claim $\pi|_D$ gives homeomorphism $\pi|_D : D \to \pi(D)$. Clearly, $\pi|_D$ is onto, continuous and open (since π is). The choice of ϵ assures $\pi|_D$ is 1:1 (bijective), hence π is homeom.

A complex atlas on X: $D_{z_0} = D(z_0, \epsilon)$, define $\varphi_{z_0} : \pi(D_{z_0}) \to D_{z_0}$ to be the inverse of $\pi|_{D_{z_0}}$. φ's are complex charts on X.

These complex charts are pairwise compatible :

$z_1, z_2 \in \mathbb{C}$, $\varphi_1 : \pi(D_{z_1}) \to D_{z_1}$, $\varphi_2 : \pi(D_{z_2}) \to D_{z_2}$, let $U = \pi(D_{z_1}) \cap \pi(D_{z_2})$

U empty - nothing to prove.
$U \neq \phi$: let $T(z) = \varphi_2(\varphi_1^{-1}(z)) = \varphi_2(\pi(z)) \quad \forall z \in \varphi_1(U)$,
? is T holomorphic on $\varphi_1(U)$?
Notice $\pi(T(z)) = \underbrace{\pi(\varphi_2}_{Id}(\pi(z))) = \pi(z) \quad \forall z \in \varphi_1(U)$

$\Rightarrow T(z) - z = \omega(z) \in L \quad \forall z \in \varphi_1(U)$; $\omega : \varphi_1(z) \to L$ is continuous & L is discrete $\Rightarrow \omega$ is locally constant on $\varphi_1(U)$ (it is constant on connected component of U.) So $T(z) = z + \omega$ for a fixed $\omega \in L \Rightarrow T$ is holomorphic. Hence $\{\varphi_z, z \in \mathbb{C}\}$ is a complex atlas on X.

C/ Graphs of holomorphic functions

$V \subseteq \mathbb{C}$. a connected open subset of $\mathbb{C}$, $g : \mathbb{C} \to \mathbb{C}$ holom. fcion on V
The graph of g: $\quad X = \{(z, g(z)) \subseteq \mathbb{C}^2 \mid z \in V\}$,
and give X the subspace topology, $\pi : X \to V$ the projection
(π is homeomorphism, its inverse sends $z \in V \mapsto (z, g(z))$)
$\Rightarrow \pi$ is a complex chart on X, whose domain covers all of X
- (complex atlas on X with single chart.)

D/ Smooth affine plane curves

A generalisation of C/ : we consider a locus $X \subseteq \mathbb{C}^2$, which is locally a graph (but perhaps not globally.) The most natural way to do this - the zero locus of a complex polynomial of two variables $f(z, w)$. For the local graph property for X, we require the Implicit Function theorem:

Theorem: Let $f(z,w) \in \mathbb{C}[z,w]$ be a polynomial, $X = \{(z,w) \in \mathbb{C}^2 \mid f(z,w) = 0\}$ be its zero locus. Let $p = (z_0, w_0)$ be a point of X (i.e., p is a root of f.) Suppose $\frac{\partial f}{\partial w}(p) \neq 0$. Then there exists a function $g(z)$ defined and holomorphic in a neighborhood of z_0, such that near p, X is equal to the graph $w = g(z)$. Moreover, $g' = -\frac{\partial f}{\partial z} \Big/ \frac{\partial f}{\partial w}$ near z_0.

Def: An affine plane curve is the zero locus in $\mathbb{C}^2$ of a polynomial $f(z,w)$. A polynomial $f(z,w)$ is non-singular at a root $p=(z_0,w_0)$ if either $\frac{\partial f}{\partial z}$ or $\frac{\partial f}{\partial w}$ is non-zero at $p \in \mathbb{C}^2$. The affine plane curve X of roots of f is non-singular at p if f is non-singular at p. The curve X is non-singular, or smooth if it is non-singular at each of its points.

Let X be a smooth affine plane curve, $f(z,w)=0$, $p=(z_0,w_0)\in X$. If $\frac{\partial f}{\partial w}\Big|_p \neq 0$, find $g_p(z)$ such that in the neighborhood U of p, X is the graph $w=g_p(z)$. The projection $\pi_z : U \to \mathbb{C}$, $(z,w)\mapsto z$ is a homeomorph. $U \to V \subseteq \mathbb{C}$ (image of this proj) $\Rightarrow$ complex chart on X.

If $\frac{\partial f}{\partial z}\Big|_p \neq 0$, do the projection $(z,w)\to w$ and build the chart based on w.

Any two charts are compatible: say, consider π_z, π_w, and $p=(z_0,w_0)$ in their common domain U. Assume, near $p\in X$, X is locally of the form $w=g(z)$ for some holomorphic function g. On $\pi_z(U)$ near z_0, π_z^{-1} sends z to $(z,g(z))$. Then the composition $\pi_w \circ \pi_z^{-1}$ sends z to $g(z)$ (i.e, it is holomorphic map.)

The connectedness assumption on X (or rather, f) is that $f(z,w)$ is an irreducible polynomial, i.e., can't be factored as $f(z,w)=g(z,w)\,h(z,w)$ for g,h non-constant polynomials.

E/ <u>Smooth projective plane curves</u>

Proj plane $\mathbb{P}^2/\mathbb{C}$ is the set of 1-dim subspaces of $\mathbb{C}^3$.
$(x,y,z)\in\mathbb{C}^3 \rightsquigarrow$ the line $[x:y:z]$, it is a point in $\mathbb{P}^2/\mathbb{C}$.
We have $[x:y:z]=[\lambda x:\lambda y:\lambda z]$ $\forall \lambda\in\mathbb{C}^*$,
so $\mathbb{P}^2/\mathbb{C} \simeq \mathbb{C}^3\setminus\{0\}/\mathbb{C}^*$ with quotient topology
for $\pi : \mathbb{C}^3\setminus\{0\} \to \mathbb{P}^2/\mathbb{C}$ $\quad [x:y:z]$ = homogeneous coordinates

(11) $\mathbb{P}^2/\mathbb{C}$ can be covered by

$U_0 = \{[x:y:z] \mid x \neq 0\}$, $U_1 = \{[x:y:z] \mid y \neq 0\}$, $U_2 = \{[x:y:z] \mid z \neq 0\}$.

$\downarrow\wr$ homeom $\quad$ $\downarrow\wr$ homeom. $\quad$ $\downarrow\wr$ homeom

$\mathbb{C}^2 \quad \mathbb{C}^2 \quad \mathbb{C}^2$

$[x:y:z] \to (y/x, z/x) \in \mathbb{C}^2$

$[1:a:b] \leftarrow (a, b)$

analogously as for U_0

F ... homogeneous polynomial of degree $d \in \mathbb{N}_0$
$\overset{\shortparallel}{F}(x, y, z)$

Because $F(\lambda x_0, \lambda y_0, \lambda z_0) = \lambda^d F(x_0, y_0, z_0)$,
so the value of F on $[\lambda x_0 : \lambda y_0 : \lambda z_0] = [x_0 : y_0 : z_0]$ is different, but whether F is zero or not is indepecedent of a representative: $X = \{[x:y:z] \mid F(x, y, z) = 0\} \subseteq \mathbb{P}^2/\mathbb{C}$.
$\overset{\cap}{\mathbb{P}^2/\mathbb{C}}$

The intersection $\underset{i=0,1,2}{X_i := X \cap U_i}$ (e.g., $X_0 = X \cap U_0 \simeq \{(a, b) \in \mathbb{C}^2 \mid F(1, a, b) \overset{\shortparallel}{} 0\}$)

is an affine plane curve in $U_i \simeq \mathbb{C}^2$, defined by a polynomial $f(a, b) = 0$ (where, for U_0 and X_0, $f(a, b) = F(1, a, b)$).

Is this under some assumption a (compact) Riemann surface?

__Def__: A homogeneous polynomial $F(x, y, z)$ is non-singular if there are no common solutions to the system
$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0 \quad \text{in } \mathbb{P}^2/\mathbb{C}.$$

__Lemma__: Suppose $F(x, y, z)$ is a degree d homogeneous polynomial. Then F is non-singular if and only if $\forall\, X_i$ is a smooth affine plane curve in $\mathbb{C}^2$.

__Pf__:

Assume one of X_i is not smooth, e.g. X_0. Define $f(u,v) = F(1,u,v)$
so X_0 is defined by $f=0$ in $\mathbb{C}^2$. X_0 is not smooth $\Rightarrow$ $\exists$ common solution
$(u_0,v_0) \in \mathbb{C}^2$ for $f=0$, $\frac{\partial f}{\partial u} = 0$, $\frac{\partial f}{\partial v} = 0$ in (u_0, v_0).
We claim that then $[1:u_0:v_0]$ is a common solution of
$F=0$, $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$ ($\Rightarrow$ F is singular). For this,
note
$$F[1:u_0:v_0] = f(u_0,v_0) = 0,$$
$$\frac{\partial F}{\partial y}[1:u_0:v_0] = \frac{\partial f}{\partial u}(u_0,v_0) = 0, \quad \text{and analogously } \frac{\partial F}{\partial z} = \dots$$
$$\frac{\partial F}{\partial x}[1:u_0:v_0] = \left(dF - u_0\frac{\partial F}{\partial y} - v_0\frac{\partial F}{\partial z}\right)[1:u_0:v_0] = 0. \quad \boxtimes$$
using the Euler formula

Because a non-singular homogeneous polynomial is automatically irreducible (a non-trivial result), so

<u>Theorem</u>: $F(x,y,z)$ - a non-singular homogeneous pol. Then the projective plane curve X (the zero locus of F in $\mathbb{P}^2$) is a compact Riemann surface. At every point of X one can take a local coordinate given by the ratios of the homogeneous coordinates.

Mention the higher dimensional case: $\mathbb{P}^n/\mathbb{C}$, finite collection of homog. pol. and their intersection.

FUNCTION SPACES AND MAPS

X - RS, $p \in X$, $f: X \to \mathbb{C}$
(defined near p, i.e. on $W \subseteq X$ open, $p \in W$.)
Any property at $p \in X$ is reflected in a local chart (and independent of its choice.)

<u>Def</u>: $f: X \to \mathbb{C}$ is a holomorphic fcion at $p \in X$ if there $\exists$ a chart $\varphi: U \to V$ with $U \subseteq X$, $p \in U$ such that $f \circ \varphi^{-1}: V \to \mathbb{C}$ is holomorphic at $\varphi(p)$. f is holom. in $W \subseteq X$ if it is holomorphic at every point of W.

Lemma: X - RS, $p \in X$, $f : \underset{X}{\overset{}{W}} \to \mathbb{C}$, $W \subseteq X$. Then

1/ f is hol. at p iff $\forall \varphi : \underset{p}{U} \to \underset{\mathbb{C}}{V}$, $f \circ \varphi^{-1}$ is hol. at $\varphi(p)$,

2/ f is hol. in $W \subseteq X$ iff $\exists$ $\{\varphi_i : U_i \to V_i\}$ with $W \subseteq \bigcup_i U_i$ s.t. $f \circ \varphi_i^{-1}$ is hol. on $\varphi_i(U_i \cap W)$ $\forall i$.

3/ f is hol. at p $\Rightarrow$ f is hol. in a neigh. of p in X.

Pf: $\left.\begin{matrix}\varphi_1 \\ \varphi_2\end{matrix}\right\}$ chart, their domain contains p, suppose $f \circ \varphi_1^{-1}$ is hol.

at $\varphi_1(p)$; is $f \circ \varphi_2^{-1}$ hol. at $\varphi_2(p)$?

$$f \circ \varphi_2^{-1} = f \circ (\varphi_1^{-1} \circ \varphi_1) \circ \varphi_2^{-1} = \underbrace{\underbrace{(f \circ \varphi_1^{-1})}_{\text{hol}} \circ \underbrace{(\varphi_1 \circ \varphi_2^{-1})}_{\text{hol}}}_{\Downarrow \ \text{hol.}}$$

The rest is straight forward. ∎

Ex: $f : X \to \mathbb{C}$ for $X = \mathbb{P}^1/\mathbb{C}$, $p = [0 : 1]$ $\left(\mathbb{C} \xrightarrow{\sim} \mathbb{C} = U_0 \subseteq \mathbb{P}^1/\mathbb{C},\ z \mapsto [z : 1]\right)$

f is holom. at $\infty := [0:1]$ iff $f(\frac{1}{z})$ is holom. at $z = 0$ $(= [1:0].)$ In part., if f is rational fcion $f(z) = \dfrac{p_1(z)}{p_2(z)}$ then f is holomorphic at ∞ iff $\deg(p_1) \leq \deg(p_2)$.

For $W \subseteq X$ open, X - RS, $\mathcal{O}_X(W)$ ... $\mathbb{C}$-algebra of hol. fcions on W: $\mathcal{O}_X(W) := \mathcal{O}(W) = \{f : W \to \mathbb{C} \mid f \text{ holomorphic}\}$.

Singularities of fctions, Meromorphic fctions

X - RS, $p \in X$, $f: \mathcal{U} \to \mathbb{C}$, $p \in \mathcal{U}$, holomorphic in $\mathcal{U} \setminus \{p\} =: \mathcal{U}^*$ (punctured neigh. of p.)

The type of singular behaviour of f at p is classified by

Def: f ... hol. fction in a punctured neigh. of $p \in X$.

A/ f has a removable singularity at p iff $\exists$ a chart $\varphi: \mathcal{U} \to V$ with $p \in \mathcal{U}$, such that $f \circ \varphi^{-1}$ has a removable singularity at $\varphi(p)$.

B/ f has a pole at p iff $\exists\ \varphi: \mathcal{U} \to V$, $p \in \mathcal{U}$, such that $(f \circ \varphi)$ has a pole at $\varphi(p)$.

C/ f has an essential singularity at p iff $\exists$ ....

Lemma: With the notation as above, f has a removable singularity (a pole, an essential singularity at $p \in X$) iff $\forall$ chart $\varphi: \mathcal{U} \to V$, $p \in \mathcal{U}$, $f \circ \varphi^{-1}$ has a removable sing. (a pole, an essential sing.) at $\varphi(p)$.

Remark: If $|f(x)|$ is bounded on $\mathcal{U}^*$, f has a removable sing at p. In this case the limit $\lim_{x \to p} f(x)$ exist, and for $f(p) := \lim_{x \to p} f(x)$ is f holomorphic at p. If $\lim_{x \to p} |f(x)| \to \infty$, f has a pole at p. If $\lim_{x \to p} |f(x)|$ does not exist, f has an essential singularity at p.

Ex: Suppose f, g are meromorphic at $p \in X$. Then $f+g$, fg and $\frac{f}{g}$ (provided $g \neq 0$) are meromorphic functions.

Ex: $\mathbb{C}/L$, L ... a lattice in $\mathbb{C}$, $\pi : \mathbb{C} \to \mathbb{C}/L$, $W \subseteq \mathbb{C}/L$ open subset, $f : W \to \mathbb{C}$ a complex-valued function; then f is meromorphic at $p \in W$ iff $\exists$ a pre-image z of p in $\mathbb{C}$ such that $f \circ \pi$ is meromorphic at z. Moreover, f is meromorphic ~~at~~ on W iff $f \circ \pi$ is meromorphic on $\pi^{-1}(W)$. Because $f \circ \pi =: g$ is L-periodic, $g(z+\omega) = g(z)$ $\forall \omega \in L$, there is bijection

$$\{\text{functions on } \mathbb{C}/L\} \overset{1:1}{\longleftrightarrow} \{\text{functions on } \mathbb{C} \text{ periodic for } L\}.$$

A meromorphic L-periodic functions on $\mathbb{C}$ is called elliptic functions, so

$$\{\text{elliptic functions on } \mathbb{C}\} \overset{1:1}{\longleftrightarrow} \{\text{meromorphic fcions on } \mathbb{C}/L\}$$

Ex: X . a projective plane curve, $F(x,y,z)=0$ for a nonsingular homogeneous polynomial F.

$G(x,y,z)$ ... hom pol. (of degree d)
$H(x,y,z)$ ... — " —

} do not vanish identically on X defined by F

Then $G/H\,\big|_X$ is a meromorphic fcion on X.

Def: X - RS, $W \subseteq X$ open subset.

$$\mathcal{M}_X(W) = \mathcal{M}(W) = \{f : W \to \mathbb{C} \mid f \text{ is meromorphic}\}.$$

X - RS, f ... hol. fcion in a punct. neighbor. $U^* = U \setminus \{p\}$ of $p \in X$, $\varphi : \underset{p \in U}{U} \to V$ a chart, $\underset{x \in U}{z = \varphi(x)}$, then

$f \circ \varphi^{-1}$ is hol. in a neigh. of $z_0 = \varphi(p) \in V \subseteq \mathbb{C}$. Therefore,

$$f(\varphi^{-1}(z)) = \sum_{n \in \mathbb{Z}} c_n (z - z_0)^n, \quad c_n \in \mathbb{C} \quad \text{(Laurent series of } f \text{ w.r. to } \varphi\text{)}$$

<u>Def</u> (the order of a meromorphic fcion)

f meromorphic at p, z ... local coordinate, its Laurent series is $\sum c_n (z - z_0)^n$. The order of f at p, $\mathrm{ord}_p(f)$, is

$$\mathrm{ord}_p(f) := \min\{n \mid c_n \neq 0\}.$$

$\mathrm{ord}_p(f)$ is independent of chart: $\psi : U \to V'$ another chart, $w := \psi(x)$ hol. coord. around p, $x \in U$; $\psi(p) = w_0$.

$T(w) = \varphi \circ \psi^{-1}$ holom. trans. fcion, then $T'(w_0) \neq 0$, so

$$z = T(w) = z_0 + \sum_{n \geq 1} a_n (w - w_0)^n, \quad a_1 \neq 0.$$

Suppose $c_{n_0}(z - z_0)^{n_0} +$ "higher order terms" ... L. series of f at p in terms of z

$c_{n_0} \neq 0 \Leftrightarrow \mathrm{ord}_p(f) = n_0$.

Because $z - z_0 = \sum_{n \geq 1} a_n (w - w_0)^n$, get

$$c_{n_0} a_1^{n_0} (w - w_0)^{n_0} + \text{"higher order terms"}$$

... L. series of f at p in terms of w

Since $c_{n_0} \neq 0$, $a_1 \neq 0 \Rightarrow$ the coeff. of $(w - w_0)^{n_0}$ is non-zero $\Rightarrow$ $\mathrm{ord}_p(f)$ is chart independent.

(17) f is holom. at p iff $\mathrm{ord}_p(f) \geq 0$, $f(p)=0$
iff $\mathrm{ord}_p(f) > 0$. f has a pole at p iff $\mathrm{ord}_p(f) < 0$,
f has neither zero nor pole iff $\mathrm{ord}_p(f) = 0$.
(f has a zero of order n at p if $\mathrm{ord}_p(f) = n \geq 1$,
pole —— " —— $= -n < 0$.)

<u>Lemma</u>: $f, g \neq 0$ meromorph. fctions at $p \in X$; then
1/ $\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$,
2/ $\mathrm{ord}_p\left(\frac{f}{g}\right) = \mathrm{ord}_p(f) - \mathrm{ord}_p(g)$,
3/ $\mathrm{ord}_p\left(\frac{1}{f}\right) = -\mathrm{ord}_p(f)$.
4/ $\mathrm{ord}_p(f \pm g) \geq \min\{\mathrm{ord}_p(f), \mathrm{ord}_p(g)\}$.

<u>Ex</u>: $f(z) = \dfrac{p(z)}{q(z)} \neq 0$ rational fction (considered as a merom. fction of $\mathbb{P}^1/\mathbb{C}$. Because
$$f(z) = c \prod_i (z - \lambda_i)^{e_i}, \quad c \in \mathbb{C}^*, \ \lambda_i \in \mathbb{C} \text{ distinct},$$
$e_i \in \mathbb{Z}$ exponents. Then $\mathrm{ord}_{z=\lambda_i}(f) = e_i \ \forall i$.
Moreover, $\mathrm{ord}_\infty(f) = \deg(q) - \deg(p) = -\sum_i e_i$.
Finally, $\mathrm{ord}_x(f) = 0$ unless $x = \infty$, $x \in \{\lambda_i\}_i$; note
$\sum_{x \in \mathbb{P}^1/\mathbb{C}} \mathrm{ord}_x(f) = 0$, a general phenomenon for merom. fctions on RS.

Let $\varphi : [z:w] \to \frac{z}{w}$ ($w \neq 0$), $\varphi^{-1}(u) = [u:1]$, to check
the fction $r([z:w]) = p(z,w)/q(z,w)$ is meromorph. on $w \neq 0$,
we must show $r \circ \varphi^{-1}$ is merom. on $\mathbb{C}$:
$$r(\varphi^{-1}(u)) = r([u:1]) = p(u,1)/q(u,1),$$
certainly merom. fction.

(18) A fūn f on X is C^∞ (smooth) at $p \in X$ if there is a chart $\varphi: U \to V$ on X with $p \in U$ such that $f \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$ (i.e., real and imaginary parts of $\mathbb{C}$-valued fūn are C^∞.) This property is again chart-independent.

Several results inherited from one complex variables in $\mathbb{C}$.

- f is meromorph. fūn on RS X; if $f \neq 0$, then the zeroes and poles of f form a discrete subset of X.
- on X compact, $f \neq 0$ has finite number of zeroes & poles.
- (the maximum modulus theorem)

  f a holomorphic fūn on a RS X; suppose $\exists$ $p \in X$: $|f(x)| \leq |f(p)|$ $\forall x \in X$. Then f is constant fūn on X.

__Ex__ (Meromorphic fūns on $\mathbb{P}^1/\mathbb{C}$ = the Riemann sphere)

We have seen that $\forall$ rational fūn $r(z) = \frac{p(z)}{q(z)}$, $p, q \in \mathbb{C}[z]$, is a meromorphic fūn. It is straightforward to prove the converse statement: Any meromorphic fūn on the Riemann Sphere is a rational fūn.

__Ex__ (Meromorphic fūns on complex ~~torus~~)

$\tau \in \mathbb{H}$ = upper half plane, $L = \mathbb{Z} + \mathbb{Z}\tau$, $X = \mathbb{C}/L$.

Analogously to $\mathbb{P}^1/\mathbb{C}$, we would expect to take the ratios of holomorphic fūns on $\mathbb{C}$ which are L-periodic; this is wrong, because there are none (i.e., just constant ones.) We introduce an analogue of linear polynomial (for $\mathbb{P}^1/\mathbb{C}$) called __theta fūn__, and build any meromorphic fūn out of it.

(19) So, for $\tau \in \mathbb{H}$ we have $\operatorname{Im}(\tau) > 0$, and define

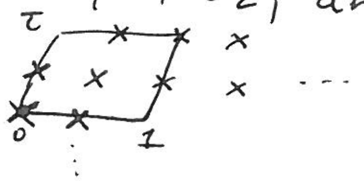
$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{i\pi(n^2\tau + 2nz)}$$

The series converges absolutely and uniformly on compact subsets of $\mathbb{C}$ $\Rightarrow$ $\theta(z)$ is analytic (and so holomorphic) function on $\mathbb{C}$.

We note $\theta(z+1) = \theta(z)$ for $\forall z \in \mathbb{C}$ (the series is the Fourier expansion of $\theta(z)$.) How $\theta(z)$ transforms under $z \mapsto z + \tau$?
An elementary computation shows

$$\theta(z+\tau) = e^{-\pi i(\tau + 2z)}\,\theta(z) \qquad \forall z \in \mathbb{C}.$$

Then z_0 is zero of θ iff $z_0 + m + n\tau$ is zero of θ for $\forall\, m,n \in \mathbb{Z}$. (moreover, the orders are the same.) One can show that the only zeroes of $\theta(z)$ are $\frac{1}{2} + \frac{1}{2}\tau + m + n\tau$, $m,n \in \mathbb{Z}$, and the zeroes are simple.



Consider the x-translate of θ:

$$\theta^{(x)}(z) := \theta\left(z - \frac{1}{2} - \frac{\tau}{2} - x\right)$$

has a simple zeroes at the points $x + L$. We notice

$$\theta^{(x)}(z+1) = \theta^{(x)}(z), \quad \theta^{(x)}(z+\tau) = -e^{-2\pi i(z-x)}\,\theta^{(x)}(z),$$

and consider the ratio

$$R(z) := \frac{\prod_i \theta^{(x_i)}(z)}{\prod_j \theta^{(y_j)}(z)}.$$

R is meromorphic on $\mathbb{C}$, it is periodic $R(z+1) = R(z)$.

The key computation is:

$$R(z+\tau) = \frac{\prod_{i=1}^{m} \theta^{(x_i)}(z+\tau)}{\prod_{j=1}^{n} \theta^{(y_j)}(z+\tau)} = (-1)^{m-n} \frac{\prod_{i=1}^{m} e^{-2\pi i(z-x_i)}\theta^{(x_i)}(z)}{\prod_{j=1}^{n} e^{-2\pi i(z-y_j)}\theta^{(y_j)}(z)}$$

$$= (-1)^{m-n} e^{-2\pi i \left[(m-n)z + \sum_j y_j - \sum_i x_i\right]} R(z)$$

and so L-invariance $\Rightarrow$

$$(-1)^{m-n} e^{-2\pi i \left[(m-n)z + \sum_j y_j - \sum_i x_i\right]} = 1 \quad \forall z \in \mathbb{C}$$

$\Rightarrow$ $m=n$ and if so, then $\left(\sum_i x_i - \sum_j y_j\right) \in \mathbb{Z}$

and so $R(z)$ descends to a meromorphic function on $\mathbb{C}/L$. In particular, $R(z)$ has (simple) zeroes at $x_i + L$ and (simple) poles at $y_j + L$ of $\mathbb{C}/L$.

HOLOMORPHIC MAPS BETWEEN RIEMANN SURFACES

We build the category of RSes. X, Y - RS

<u>Def</u>: A mapping $F: X \to Y$ is holom. at $p \in X$ iff $\exists$ charts $\varphi_1 : \mathcal{U}_1 \to V_1$, $\varphi_2 : \mathcal{U}_2 \to V_2$ with $p \in \mathcal{U}_1$, $V_1 \subset \mathbb{C}$ on X and $F(p) \in \mathcal{U}_2$, $V_2 \subset \mathbb{C}$ on Y

such that $\varphi_2 \circ F \circ \varphi_1^{-1}$ is holomorphic at $\varphi_1(p)$.
$\mathbb{C} \to \mathbb{C}$

If F is defined on an open $W \subseteq X$, F is holom. on W if F is holom. at $\forall$ point of W. F is holom. map if F is holom. on X.

Holomorphicity can be checked with any pair of charts.

<u>Lemma</u>: X, Y - RS, $F: X \to Y$ holom. map.

1/ F is holom. iff $\forall$ pair of local charts φ_1, φ_2 as above $\varphi_2 \circ F \circ \varphi_1^{-1}$ is holom.

2/ Analogously for $W \subseteq X$ open, and a covering $\bigcup_i \mathcal{U}^{(i)} \supseteq W, \dots$

(21)

Holomorphic maps behave quite well with respect to ~~#~~ composition:

<u>Lemma</u>: 1/ If F is holomorphic, F is continuous and C^∞.

2/ The composition of holomorphic maps is holomorphic: if $F: X \to Y$ and $G: Y \to Z$ are holom. maps, then $G \circ F: X \to Z$ is a holom. map.

3/ The composition of a holomorphic map with a holomorphic function is holomorphic: $F: X \to Y$ holomorphic, $g: W \to \mathbb{C}$ holom. map, then $g \circ F$ is a holom. fcion on $F^{-1}(W)$. ($W \subseteq Y$)

4/ Analogously to 3/, the composition of a holom. map and a meromorphic fcion is a meromorphic fcion.

<u>Ex</u>: The identity map $Id: X \to X$ is holomorphic for any Riemann surface X.

$\Rightarrow$ Riemann surfaces form a category

The previous properties can be formulated: $F: X \to Y$, $W \subseteq Y$ open, F induces $\mathbb{C}$-algebra homom.:

$$F^*: \mathcal{O}_Y(W) \to \mathcal{O}_X(F^{-1}(W))$$
$$g \mapsto F^*(g) = g \circ F$$

(the same for meromorphic fcions.)

We have $F^* \circ G^* = (G \circ F)^*$ for $F: X \to Y$, $G: Y \to Z$.

<u>Def</u>: An isomorphism (or, biholomorphism) between RS X, Y is a holomorphic map $F: X \to Y$ which is bijective and $F^{-1}: Y \to X$ is holomorphic. A biholomorphic map $F: X \to X$ is called an automorphism of X. If there $\exists$ an isomorphism $F: X \to Y$, we say X, Y are isomorphic (biholomorphic).

<u>Ex</u>: The Riemann sphere $S^2 \simeq \mathbb{C} \cup \{\infty\}$ and the projective line $\mathbb{P}^1/\mathbb{C}$ are isomorphic via

$$\mathbb{P}^1/\mathbb{C} \ni [z; w] \mapsto \left(2\mathrm{Re}(z\bar{w}), 2\mathrm{Im}(z\bar{w}), \frac{|z|^2 - |w|^2}{|z|^2 + |w|^2}\right) \in S^2 \subseteq \mathbb{R}^3.$$

Some topological properties of holomorphic maps:

Theorem (Open mapping theorem):

$F: X \to Y$ a non-constant holomorphic map between RS X, Y.
Then F is an open mapping.

Proposition: Let X be a compact RS, $F: X \to Y$ be a nonconstant holom. map into RS Y. Then Y is compact and F is onto.

Pf: F is holom. and $X \subseteq X$ is open $\Rightarrow F(X)$ is open in Y (open map. th.) X is compact & F is holom. $\Rightarrow F(X)$ is compact. Y is Hausdorff $\Rightarrow F(X)$ is closed in Y. Hence $F(X)$ is both closed & open in Y $\underset{Y \text{ is connected}}{\Rightarrow}$ $F(X) = Y$. Thus F is onto and Y is compact.

The pre-image $F^{-1}(y)$, $y \in Y$, $F: X \to Y$, is a discrete subset of X (in particular, if X, Y are compact RS, $F^{-1}(y)$ is a non-empty finite set $\forall y \in Y$.)

Global properties of holomorphic maps ← will study certain functions of local invariants.

Prop. (Local normal form of a holom. map) $F: X \to Y$ holom. map, defined at $p \in X$, not constant. Then there $\exists!$ $m \geq 1$ integer, satisfying: $\forall$ $\varphi_2: U_2 \to V_2$ on V, centered at $F(p)$, $\exists$ a chart $\varphi_1: U_1 \to V_1$ on X centered at p such that
$$\varphi_2\left(F\left(\varphi_1^{-1}(z)\right)\right) = z^m.$$

Pf: Fix a chart φ_2 on Y centered at $F(p)$, choose any chart $\psi: U \to V$ on X centered at p. Then the Taylor series of $T := \varphi_2 \circ F \circ \psi$, $w \mapsto T(w)$, is of the form
$$T(w) = \sum_{i=m}^{\infty} c_i w^i, \quad c_i \in \mathbb{C}, \ c_m \neq 0, \ m \geq 1$$
since $T(0) = 0$.

(23) Thus $T(w) = w^m S(w)$, where $S(w)$ is holomorphic at $w=0$ and $S(0) \neq 0$. Then there is a function $R(w)$ holom. at $w=0$ and $R(w)^m = S(w)$, i.e., $T(w) = (w\, R(w))^m$. Define $\eta(w) = w\, R(w)$; since $\eta'(0) \neq 0$, η is invertible near $w=0$ invertible by implicit function theorem and is also holom. Hence $\varphi_1 := \eta \circ \psi$ is also a chart on X, defined and centered near p. Think of η a giving the new coordinate z $(z := \eta(w))$, we see z, w are related by $z = w\, R(w)$. Then

$$
\begin{aligned}
(\varphi_2 \circ F \circ \varphi_1^{-1})(z) &= (\varphi_2 \circ F \circ \psi^{-1} \circ \eta^{-1})(z) = \\
&= (T \circ \eta^{-1})(z) \\
&= T(w) \\
&= (w\, R(w))^m \\
&= z^m.
\end{aligned}
$$

The exponent m can be detected by ~~...~~ studying the topological properties of F near p, and is independent of the choices of the local coordinates => uniqueness.

<u>Def</u>: The multiplicity of F at p, $\mathrm{mult}_p(F)$, is the unique integer m such that there are local coord. near p, $F(p)$, $F: z \mapsto z^m$.

We have $\mathrm{mult}_p(F) \geq 1$, and for $\varphi: U \to V \subseteq \mathbb{C}$ a chart map for X, considered as a holomorphic map $X \to \mathbb{C}$. Then φ has multipl. one at $\forall$ point of U.

A way to compute $\mathrm{mult}_p(F)$ without introducing a chart with local normal form for F: pick z ... chart for p, $p \mapsto z_0$
w ... — " — $F(p)$, $F(p) \mapsto w_0$

Then F corresponds to $w = h(z)$ for h holom., $w_0 = h(z_0)$

Then the multiplicity $\mathrm{mult}_p(F)$ of F at p is given by
$$\mathrm{mult}_p(F) = 1 + \mathrm{ord}_{z_0}\left(\frac{dh}{dz}\right);$$
if $h(z) = h(z_0) + \sum_{i=m}^{\infty} c_i (z-z_0)^i$, $c_i \in \mathbb{C}$, $m \geq 1$ and $c_m \neq 0$, then $\mathrm{mult}_p(F) = m$.

This shows that the points of the domain where F has multiplicity at least 2 form a discrete set.

Def: $F: X \to Y$ a non-constant hol. map. A point $p \in X$ is a ramification point for F if $\mathrm{mult}_p(F) \geq 2$. A point $y \in Y$ is a branch pt for F if it is the image of a ramification point for F.

Ex: (Smooth plane curves) X... smooth affine plane curve, defined by $f(x,y) = 0$. Define $\pi: X \to \mathbb{C}$, $(x,y) \mapsto x$. Then π is ramified at $p \in X$ iff $\left(\frac{\partial}{\partial y} f\right)(p) = 0$.

Let X be a smooth projective plane curve, defined by homog. polyn. $F(x,y,z) = 0$; Consider the map $G: X \to \mathbb{P}^1_{\mathbb{C}}$, $[x:y:z] \mapsto [x:z]$.

Then G is ramified at $p \in X$ iff $\frac{\partial F}{\partial y}(p) = 0$.

The relationship between the multiplicity (defined for a holomorphic map between Riemann surfaces) and the order (defined for a meromorphic function) is the content of

Lemma: Let f be a meromorphic function on a RS X, with $F: X \to \mathbb{C}_\infty \simeq \mathbb{C} \cup \{\infty\}$ the associated holomorphic map.

(1) If $X \ni p$ is a zero of f, then $\mathrm{mult}_p(F) = \mathrm{ord}_p(f)$.

(2) If p is a pole of f, then $\mathrm{mult}_p(F) = -\mathrm{ord}_p(f)$.

(3) If p is neither a zero nor a pole of f, $\mathrm{mult}_p(F) = \mathrm{ord}_p(f - f(p))$.

(the proof follows from the observation on the top of this page)

The degree of a holomorphic map between RS:

Prop: let $F: X \to Y$ be a non-constant holomorphic map between compact RS. $\forall y \in Y$, define $d_y(F)$ to be

$$d_y(F) = \sum_{p \in F^{-1}(y)} \text{mult}_p(F).$$

Then $d_y(F)$ is constant, independent of y.

Pf: Show the function $y \mapsto d_y(F)$ is locally constant, and if Y is connected, it is constant.

Consider $D = \{ z \in \mathbb{C} \mid \|z\| < 1 \}$ and the map $f: D \to D$, $z \mapsto f(z) = z^m$ for some $m \geq 1$; f is holom. and onto, the only ramif. point is at $z=0$, where the multipl. is m. All other points have mult 1. $\forall w \in D$, $w \neq 0$, f has m-preimages (the m m-th roots of w) each of mult. $= 1$; if $w=0$, its preimage $z=0$ has multiplicity $m \Rightarrow$ in this (local nomal) chart is mult_p constant.

Fix $y \in Y$, $\{x_1, .., x_n\}$ be the inverse image of ~~y~~ under F. w... a complex chart on Y, centered at y. Local nomal form $\Rightarrow$ $\exists$ coordinates z_i on X, z_i centered at x_i $\forall i = 1, \dots, n$, such that in a neigh. of x_i the map F sends z_i to ~~////~~ $w = z_i^{m_i}$. This gives disjoint union description of F.

By compactness of X, near y there are no other preimages left unaccounted for which are not in the neighborhoods of the $x_1, \dots, x_n$.

Def: $F: X \to Y$ a non-constant holom. map RS X, Y. The degree of F, $\deg(F)$, is the integer $d_y(F)$ $\forall y \in Y$.

A holomorphic map between compact RS is an isomorphism iff it has degree one. Deleting branch points (in Y) of F, and all of their pre-images in X, we obtain a covering map

$$\begin{array}{ccc} F: U & \to & V \\ \cap & & \cap \\ X & & Y \end{array}$$

in the sense of topology: $\forall x \in V$ $\exists$ $N \subseteq V$ open, $x \in N$

such that $F^{-1}(N) = \dot{\bigcup} M_i$, $M_i \subseteq U$: $F|_{M_i}$ is a homeom.
disjoint
open union

of M_i with N.

Prop. f.. non-constant meromorphic function on a compact RS. Then
$$\sum_{p \in X} \mathrm{ord}_p(f) = 0.$$

Pf: $F : X \to \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ associated holom. map to the Riemann sphere. Let $\{x_i\}_i$ be the points mapping to 0, $\{y_j\}_j$ be the points mapping to ∞ (x_i are the zeroes of f, y_j are the poles of f.) Let d be the degree of F.

By the definition of degree,
$$d = \sum_i \mathrm{mult}_{x_i}(F), \quad d = \sum_j \mathrm{mult}_{y_j}(F).$$
The only points of X, where f has non-zero order, are its zeroes and poles, $\{x_i\}_i$ & $\{y_j\}_j$. We have
$$\mathrm{mult}_{x_i}(F) = \mathrm{ord}_{x_i}(f), \quad \mathrm{mult}_{y_j}(F) = -\mathrm{ord}_{y_j}(f)$$
Hence
$$\begin{aligned} \sum_p \mathrm{ord}_p(f) &= \sum_i \mathrm{ord}_{x_i}(f) + \sum_j \mathrm{ord}_{y_j}(f) \\ &= \underbrace{\sum_i \mathrm{mult}_{x_i}(F)}_{= d} - \underbrace{\sum_j \mathrm{mult}_{y_j}(F)}_{= d} \\ &= 0. \end{aligned}$$

Ex: Any meromorphic function on a complex tori is a ratio of translated theta functions. Denoting $\{p_i\}_i$ the zeroes and $\{q_j\}_j$ the poles of a meromorphic function. Then the previous global constraint $\sum_{p \in X} \mathrm{ord}_p(f) = 0$ for X a complex torus reduces to $\sum_i p_i = \sum_j q_j \mod L$.

The Euler number of a compact surface

S ... compact 2-dim manifold $\supseteq$ cpt. Riemann surface
A triangulation of S is a decomposition of S into closed subsets, homeomorphic to a triangle and such that any two triangles are either disjoint, or meet at a single vertex or meet along a single edge.

Def: Let S be a compact 2-man.; suppose a triangulation is given, v - vertices. Then the Euler number of S
e - edges
t - triangles

w. r. to this triangulation is $e(S) = v - e + t$.

A central result of algebraic topology states:

Proposition: The Euler number is independent of triangulation. For a compact orientable 2-manifold (i.e., a Riemann surface) without boundary of topological genus g, the Euler number is $2-2g$.

Pf: It is called on the notion of refinement of given triangulation and ~~ths~~ the independence of $e(S)$ on its choice.

A relation between Euler characteristics of X, Y and the holomorphic map $f: X \to Y$ is known as the Hurwitz formula:

Theorem (Hurwitz's formula) X, Y - RS, $F: X \to Y$ a non-constant hol. map. Then

$$2g(X) - 2 = \deg(F)\,(2g(Y)-2) + \sum_{p\in X} (\operatorname{mult}_p(F) - 1).$$

Pf: X .. compact RS $\Rightarrow$ # ram. points is finite, so the sum on the RHS is finite.
Triangulation of Y, such that $\forall$ branch point is a vertex of Y. v, e, t - triangulation of Y

lift the triang. via F to X, assume e', v', t' triang. on X,
$\forall$ ramif. point of F is a vertex on X.

There are no ram. points over general pts of any triangle,
$\forall$ ~~tri~~ lifts to $\deg(F)$ triangles ~~bf~~ in X. So $t' = \deg(F)\, t$.
Similarly, $e' = \deg(F)\, e$.

Fix a vertex $q \in Y$; the number of preimages of q in X is
$|F^{-1}(q)|$:

$$|F^{-1}(q)| = \sum_{p \in F^{-1}(q)} 1 = \deg F + \sum_{p \in F^{-1}(q)} (1 - \mathrm{mult}_p(F)).$$

Therefore,

$$v' = \sum_{\substack{\text{vertex} \\ q \text{ of } Y}} \left[\deg(F) + \sum_{p \in F^{-1}(q)} (1 - \mathrm{mult}_p(F)) \right]$$

$$= \deg(F)\, v - \sum_{\substack{\text{vertex} \\ q \text{ of } Y}} \sum_{p \in F^{-1}(q)} (\mathrm{mult}_p(F) - 1)$$

$$= \deg(F)\, v - \sum_{\substack{\text{vertex} \\ p \text{ of } X}} (\mathrm{mult}_p(F) - 1),$$

and so

$$2g(X) - 2 = -e(X) = -v' + e' - t' =$$

$$= -\deg(F)\, v + \sum_{\substack{\text{vertex } p \\ \text{of } X}} (\mathrm{mult}_p(F) - 1) + \deg(F)\, e - \deg(F)\, t$$

$$= -\deg(F)\, e(Y) + \sum_{\substack{\text{vertex } p \\ \text{of } X}} (\mathrm{mult}_p(F) - 1)$$

$$= \deg(F)\,(2g(Y) - 2) + \sum_{p \in X} (\mathrm{mult}_p(F) - 1).$$

$\square$

Integration on RS

Need to have objects to integrate $\rightsquigarrow$ diff. forms

Def: A holom. 1-form on an open $V \subseteq \mathbb{C}$ is an expression $\omega := f(z)\,dz$, where f is a holom. fun on V; ω is a holom. 1-form in the complex variable z.

Def:
$$\left.\begin{array}{l} \omega_1 = f(z)\,dz \ \ldots \text{ hol. 1-form in } z \text{ on } V_1 \subseteq \mathbb{C} \\ \omega_2 = g(w)\,dw \ \ldots \ \text{—"—} \ w \text{ on } V_2 \subseteq \mathbb{C} \end{array}\right\} \quad T: w \mapsto T(w) = z \text{ hol. map } V_2 \to V_1$$

Then ω_1 transforms to ω_2 under T if $g(w) = f(T(w))\,T'(w)$.
($\Leftrightarrow$ $dz = T'(w)\,dw$)

Def: X ... RS, a hol. 1-form on X is a collection of hol. 1-forms $\{\omega_\varphi\}_\varphi$, one for each chart $\varphi: U \to V$ in the coordinate for V, such that for
$$\left.\begin{array}{l} \varphi_1: U_1 \to V_1 \\ \varphi_2: U_2 \to V_2 \end{array}\right\} \text{ overlap, i.e. } U_1 \cap U_2 \neq \emptyset$$
then ω_{φ_1} transforms to ω_{φ_2} under $T = \varphi_1 \circ \varphi_2^{-1}$.

In fact, one does not need to define 1-form on every chart, just on charts on some atlas.
Analogously, we have

Def: A meromorphic 1-form on $V \subseteq \mathbb{C}$ is $\omega = f(z)\,dz$, where f is meromorphic on V (in the coordinate z)

The definition on the intersection and on a Riemann surface is the same as for the holomorphic 1-forms.
Let ω be a merom. 1-form, defined around $p \in X$, $\omega = f(z)\,dz$ for complex coord. z, f meromorp. at $z = 0$.

Def: The order of ω at p, $\mathrm{ord}_p(\omega)$, is the order of f at p, $\mathrm{ord}_p(f)$.

Notice $\mathrm{ord}_p(\omega)$ is independent of the choice of complex chart.

(30) ω - mer. 1-form;

$\mathrm{ord}_p(\omega) = n > 0$ ... p is a zero of order n (for ω)

$\mathrm{ord}_p(\omega) = -n < 0$ ... — " — pole — " —

As for the description of 1-forms, we use a single formula in a specific chart and then transform it to the whole X. Note - a meromorphic form on U does not need to extend to a ~~holom~~ meromorphic form on X, and does not need to extend uniquely.

$e^z dz$ is hom. on $\mathbb{C}$,
does not extend as
a holomorphic 1-form
on $\mathbb{P}^1\mathbb{C} \supseteq \mathbb{C}$.

$f \in C^\infty(V)$ ... smooth fcion on V; f is holom. iff $\frac{\partial f}{\partial \bar{z}} = 0$.

$x, y, z, \bar{z}$, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ $\quad dz = dx + i dy$

$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ $\quad d\bar{z} = dx - i dy$

<u>Def</u>: C^∞ 1-form on $V \subseteq \mathbb{C}$ is an expression of the form

$$\omega = f(z,\bar{z})\,dz + g(z,\bar{z})\,d\bar{z}$$

for f, g C^∞-fcions on V.

The transformation rule is the following:

<u>Def</u>: $\omega_1 = f_1(z,\bar{z})\,dz + g_1(z,\bar{z})\,d\bar{z}$ is a C^∞ 1-form in the coordinate z, $V_1 \subseteq \mathbb{C}$ open

$\omega_2 = f_1(w,\bar{w})\,dw + g_1(w,\bar{w})\,d\bar{w}$ — " —

w, $V_2 \subseteq \mathbb{C}$ open.

$V_2 \xrightarrow{T} V_1$ a holom. map ; then ω_1 transforms to ω_2
$w \mapsto z = T(w)$ under T iff

$f_2(w,\bar{w}) = f_1(T(w), \overline{T(w)})\,T'(w)$

$g_2(w,\bar{w}) = g_1(T(w), \overline{T(w)})\,\overline{T'(w)}$

The reason for the last Def: if $z = T(w)$, then $dz = T'(w)\,dw$, $d\bar{z} = \overline{T'(w)}\,d\bar{w}$

$\begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} dw \\ d\bar{w} \end{pmatrix}$, no mixing in hol. & anti-hol

(31)

Def: X – RS, a C^∞- 1-form on X is a collection of C^∞ 1-forms $\{\omega_\varphi\}_\varphi$, $\varphi: U \to V \subseteq \mathbb{C}$ a chart;

$$\left.\begin{aligned} \varphi_1 &: U_1 \to V_1 \\ \varphi_2 &: U_2 \to V_2 \end{aligned}\right\} U_1 \cap U_2 \neq \emptyset$$

, then C^∞ 1-form ω_{φ_1} transforms to ω_{φ_2} under $T = \varphi_1 \circ \varphi_2^{-1}$.

The holomorphic transformations preserve $\left.\begin{matrix} dz \\ d\bar{z}\end{matrix}\right\}$ parts of C^∞ 1-form on X.

Def: A C^∞ 1-form is of type $\begin{matrix}(1,0)\\(0,1)\end{matrix}$ if it is locally of the form $\begin{matrix} f(z,\bar{z})\,dz \\ g(z,\bar{z})\,d\bar{z}\end{matrix}$.

The type $\begin{matrix}(1,0)\\(0,1)\end{matrix}$ is preserved by transform given by transition fcion $\Rightarrow$ chart independent

To integrate over a surface, we need

Def: A C^∞ 2-form on $V \subseteq \mathbb{C}$ is an expression $\eta = f(z,\bar{z})\,dz \wedge d\bar{z}$ for f a C^∞-fcion on V (written in the coordinate z for V.)

Suppose

$$\left.\begin{aligned} \eta_1 &= f(z,\bar{z})\,dz\wedge d\bar{z} \quad C^\infty \text{ 2-form in } z \text{ on } V_1 \\ \eta_2 &= g(w,\bar{w})\,dw\wedge d\bar{w} \quad \text{— '' —} \quad w \quad V_2 \end{aligned}\right\} \begin{matrix} w \to z := T(w) \\ \text{holom. trans.} \\ V_2 \to V_1 \end{matrix}$$

Then we say $\eta_1 \to \eta_2$ under T if $g(w,\bar{w}) = \|T'(w)\|^2 f(T(w),\overline{T(w)})$ with $\|T'(w)\|^2 = T'(w)\overline{T'(w)}$.

The transport of this concept to a RS amounts to

Def: X – RS, C^∞ 2-form on X is a collection of C^∞ 2-forms $\{\eta_\varphi\}$, $\varphi: U \to V \subseteq \mathbb{C}$, such that $\varphi_i : U_i \to V_i$, $i=1,2$, on $U_1 \cap U_2$ transform η_{φ_1} to η_{φ_2} under $T = \varphi_1 \circ \varphi_2^{-1}$.

There are several operations on differential forms:

ω … C^∞ 1-form, $h \in C^\infty(X)$, $h\omega$ … 1-form on X defined by $\omega = f\,dz + g\,d\bar{z}$, $h\omega = hf\,dz + hg\,d\bar{z}$ locally, gives well-defined global 1-form on X.

If ω is $(1,0)$ $\Rightarrow$ $h\omega$ is $(1,0)$ 1-form,
$(0,1)$ $\Rightarrow$ — '' — $(0,1)$ — '' —

If ω is holom., h is holom. $\Rightarrow$ $h\omega$ is holom.

If h, ω are meromorphic at $p \in X \Rightarrow \mathrm{ord}_p(h\omega) = \mathrm{ord}_p(h) + \mathrm{ord}_p(\omega)$.

Analogously for a C^∞ - 2-forms

$f \in C^\infty(X)$, then there are C^∞- 1-forms $df, \partial f, \bar\partial f$ on X by

$\partial f = \frac{\partial f}{\partial z} dz$, $\bar\partial f = \frac{\partial f}{\partial \bar z} d\bar z$, $df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z$ in $\varphi: U \to V \subseteq \mathbb{C}$ a complex chart with coordinate z

C^∞- function is holomorphic iff $\bar\partial f = 0$, there are $d, \partial, \bar\partial$ - Leibnitz rules.

C^∞ 1-form ω is exact on an open $U \subseteq X$ if $\exists$ C^∞-fun f: $\omega = df$.

ω_1, ω_2 - C^∞ 1-forms on X, in the local variable z, $\left.\begin{array}{l}\omega_1 = f_1 dz + g_1 d\bar z \\ \omega_2 = f_2 dz + g_2 d\bar z\end{array}\right\} \Rightarrow$

the wedge product $\omega_1 \wedge \omega_2 = (f_1 g_2 - f_2 g_1) dz \wedge d\bar z$ defines a well-define C^∞ 2-form on X.

Analogously, C^∞ 1-form on X gives C^∞ 2-form $d\omega, \bar\partial\omega, \partial\omega$ on X by

$\partial\omega = \frac{\partial g}{\partial z} dz \wedge d\bar z$, $\bar\partial \omega = -\frac{\partial f}{\partial \bar z} dz \wedge d\bar z$, $d\omega = \left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar z}\right) dz \wedge d\bar z$

in the local chart z, $\varphi: U \to V \subseteq \mathbb{C}$. The 2-form is well-defined,

a C^∞ 1-form is holomorphic if $\bar\partial\omega = 0$. There is ~~the~~ a Leibnitz rule

$d(f\omega) = df \wedge \omega + f\, d\omega$, $\partial(f\omega) = \partial f \wedge \omega + f \partial\omega$, $\bar\partial(f\omega) = \bar\partial f \wedge \omega + f\bar\partial$

Notice $ddf = \partial\partial f = \bar\partial\bar\partial f = 0$, $\partial\bar\partial f = -\bar\partial\partial f$ for any C^∞ 1-form.

A C^∞ function f is harmonic on $U \subseteq X$ iff $\partial\bar\partial f = 0$ on U.

A C^∞ 1-form is d-closed iff $d\omega = 0$

— " — ∂ - closed — " — $\partial\omega = 0$ — " —

$\bar\partial$ - closed $\bar\partial \omega = 0$

$ddf = 0 \Rightarrow$ (exact 1-form $\overset{\Rightarrow}{\not\Leftarrow}$ closed 1-form)

Cauchy - Riemann conditions $\Rightarrow$ A C^∞ 1-form of type $(1,0)$ is holomorp. iff $\bar\partial\omega = 0$

(33) X, Y - RS, $F: X \to Y$ hol. map, ω a C^∞ 1-form on Y

$(F^*\omega)$ —"— on X defined by the rule:

$\varphi: U \to V$ on X
$\psi: U' \to V'$ on Y } complex charts on X resp. Y

F is given by hol. func h
$w \to h(w) =: z$

$$\omega = f(z,\bar z)\,dz + g(z,\bar z)\,d\bar z \mapsto (F^*\omega) = f(h(w), \overline{h(w)})\,h'(w)\,dw + g(h(w), \overline{h(w)})\,\overline{h'(w)}\,d\bar w$$

This gives a well-defined C^∞ 1-form $F^*\omega$ on X, the pull-back of ω.

ω holom. $\Rightarrow$ $F^*\omega$ holom., ω merom. $\Rightarrow$ $F^*\omega$ merom., $\omega^{(1,0)}_{(0,1)} \Rightarrow F^*_\omega{}^{(1,0)}_{(0,1)}$

The same concept applies to C^∞ 2-forms. We have

$F^*(df) = d(F^*f)$, the same for $d \to \partial, \bar\partial$
$f \to \omega$

<u>Lemma</u>: $F: X \to Y$ a holom. map, ω - merom. 1-form. For $p \in X$ we have

$$\mathrm{ord}_p(F^*\omega) = (1 + \mathrm{ord}_{F(p)}\,\omega)\,\mathrm{mult}_p(F) - 1$$

<u>Pf</u>: Complex chart w at p, z at $F(p)$, near p has F the form $w \to z = w^n$, $n = \mathrm{mult}_p(F)$. In z,

$c \in \mathbb{C}^*$ $\omega = (cz^k + \text{high. order terms in } z)\,dz$, $k = \mathrm{ord}_{F(p)}(\omega)$. Thus

$F^*\omega = (c\,w^{nk} +$ —"— $w)\,dw\,(n w^{n-1})$ in w, so

$\mathrm{ord}_p(F^*\omega) = nk + n - 1$, as claimed. $\square$

$U \underset{\text{open}}{\subseteq} X$, the notation $\mathcal{E}(U)$, $\mathcal{E}^{(1)}(U)$, $\mathcal{E}^{(1,0)}(U)$, $\mathcal{E}^{(0,1)}(U)$, $\mathcal{E}^{(2)}(U)$, $\mathcal{O}(U)$, $\Omega^1(U)$, $\mathcal{M}(U)$, $\mathcal{M}^1(U)$,

for $\mathbb{C}$- vector space ($\mathcal{E} = C^\infty$; $\mathcal{O}, \Omega^1 \equiv$ holomorphic, $\mathcal{M} \equiv$ merom.)

<u>Def</u>: The integral of ω along $\gamma: [a,b] \to X$ (a path in X) is defined by

$$\int_\gamma \omega = \sum_i \int_{t=a_{i-1}}^{a_i} \left[f_i(z(t), \overline{z(t)})\,z'(t) + g_i(z(t), \overline{z(t)})\,\bar z'(t)\right] dt$$

with the partition $\{\gamma_i\}_i$ of γ so that $\forall i$ γ_i is C^∞ on its domain $[a_{i-1}, a_i]$ and its image is contained in (φ_i, U_i).

If the image of γ is contained in a single chart $\varphi: U \to V \subseteq \mathbb{C}$, $\omega = f\,dz + g\,d\bar z$, then $\int_\gamma \omega = \int_{\varphi\circ\gamma} f\,dz + g\,d\bar z$.

(34) It can be immediately verified:

(1) The integral is independent of parametrization: $\int_{\gamma\alpha} \omega = \int_{\gamma} \omega \quad \forall \alpha \in Diff([a,b])$

(2) The integral — " — $\mathbb{C}$-linear: $\int_{\gamma} (\lambda_1 \omega_1 + \lambda_2 \omega_2) = \lambda_1 \int_{\gamma} \omega_1 + \lambda_2 \int_{\gamma} \omega_2 \quad \forall \lambda_1, \lambda_2 \in \mathbb{C}$

(3) For $f \in C^\infty(U_\gamma)$, $\int_{\gamma} df = f(\gamma(b)) - f(\gamma(a))$.

a neigh. containing γ
$U_\gamma \subseteq X$

(4) For $\gamma = \sum \gamma_i$, $\int_{\gamma} \omega = \sum_i \int_{\gamma_i} \omega$.

(5) $\int_{-\gamma} \omega = - \int_{\gamma} \omega$

(6) $F: X \to Y$ holom. map, $F_* \gamma := F \circ \gamma$, then

$$\int_{F_* \gamma} \omega = \int_{\gamma} F^* \omega.$$

A chain on RS X is a finite formal sum of paths, with $\mathbb{Z}$-coefficients. Then for a chain $\gamma = \sum_j n_j \gamma_j$, $n_j \in \mathbb{Z}$, and C^∞ 1-form ω:

$$\int_{\gamma} \omega = \sum_j n_j \int_{\gamma_j} \omega.$$

Let ω be meromorphic at a point $p \in X$. A local coordinate z centered at p allows to write ω as a Laurent series:

$$\omega = f(z)\,dz = \left(\sum_{n=-M}^{\infty} c_n z^n\right) dz \quad , \; c_{-M} \neq 0 \; , \; \mathrm{ord}_p(\omega) = -M$$

<u>Def</u>: The residue of ω at p, $Res_p(\omega)$, is the coeff. c_{-1} of a Laurent series for ω at p.

For $\omega \in \mathcal{M}^1(U_p)$ (an open neigh. containing p), γ a small path in X enclosing p ("contractible") (and not enclosing any other pole of ω), holds $Res_p(\omega) = \frac{1}{2\pi i} \int_{\gamma} \omega$.

The proof goes through by residue theorem in $\mathbb{C}$, because ~~...~~, a complex chart can be chosen containing γ. The definition is chart-independent.

Lemma: f a meromophic fcion at $p \in X$. Then $\frac{df}{f}$ is a meromorphic 1-form at p, and $Res_p\left(\frac{df}{f}\right) = ord_p(f)$.

Pf: In a chart centered at p with local coordinate z, assume $ord_p(f) = n$. Then $f = cz^n + \dots$ near p, $c \neq 0$. We have $f^{-1} = c^{-1}z^{-n}$ + higher order terms near $p \in X$, and so $df = (ncz^{n-1}$ + higher order terms$)dz$ near p; finally,

$$\frac{df}{f} = (nz^{-1} + \text{higher order terms})dz \Rightarrow Res_p\left(\frac{df}{f}\right) = n = ord_p(f).$$

□

As for the integration of 2-forms, we fix $T \subseteq X$ a triangle in X, contained in a domain of chart $\varphi: U \to V \subseteq \mathbb{C}$. For η a C^∞ 2-form on X, $\eta = f(z,\bar{z})\,dz \wedge d\bar{z}$ in (U,φ), we define

$$\iint_T \eta = \iint_{\varphi(T)} f(z,\bar{z})\,dz \wedge d\bar{z} = \underbrace{\iint_{\varphi(T)} (-2i) f(x+iy, x-iy)\,dx \wedge dy}_{\text{standard surface integral over a domain in } \mathbb{C} \cong \mathbb{R}^2}$$

Taking a 2-chain and refining it in such a way that $\forall$ triangle (= 2-simplex) lies in a complex chart, we arrive at Stokes theorem (for D a triangulable closed subset of a RS X): for ω a C^∞ 1-form, $\int_{\partial D} \omega = \iint_D d\omega$.

Theorem (The Residue theorem):

ω - meromorphic 1-form on compact RS X. Then

$$\sum_{p \in X} Res_p(\omega) = 0.$$

Pf: $\{p_1, \dots, p_n\}$ poles of ω on X. $\forall p_i$, let γ_i be a small path enclosing p_i, let U_i be its interior ($p_i \in U_i$, and no other pole is in U_i);

$$\int_{\gamma_i} \omega = 2\pi i \, Res_{p_i}(\omega).$$

For $D = X - \dot{\bigcup_i} U_i$, then D is triangulable

(36) and $\partial D = -\sum_i \gamma_i$ as a chain on X $\Rightarrow$

$$\sum_i \mathrm{Res}_{p_i}(\omega) = \frac{1}{2\pi i}\sum_i \int_{\gamma_i} \omega = -\frac{1}{2\pi i}\int_{\sum_i \gamma_i} \omega = -\frac{1}{2\pi i}\int_{\partial D} \omega \overset{\text{Stokes th.}}{=} -\frac{1}{2\pi i}\iint_D \underbrace{d\omega}_{=0} = 0$$

because ω is holomorphic in a neigh. of D. $\square$

Applying the Residue theorem to $\frac{df}{f}$, we get

<u>Corollary</u>: $f \in \mathcal{M}^*(X)$ on compact RS X. Then

$$\sum_{p\in X} \mathrm{ord}_p(f) = 0.$$

Divisors & Meromorphic functions

X - RS, $\mathbb{Z}^X := \{f : X \to \mathbb{Z}\}$

$(f_1 + f_2)(p) := f_1(p) + f_2(p)$ ab. group

$D \in \mathbb{Z}^X$, $\operatorname{Sup}(D) := \{p \in X \mid D(p) \neq 0\}$
support of D

Def: A divisor on X is $D : X \to \mathbb{Z}$, $\operatorname{Sup}(D)$ is a discrete subset of X. The set of divisors form an abelian group under pointwise addition.

X - compact $\Rightarrow$ $\operatorname{Sup}(D)$ finite ($\mathbb{Z}^X \equiv$ free abelian group on points of X)

$$D = \sum_{p \in X} D(p) \cdot p \qquad (\underset{\mathbb{Z}}{D(p)} \neq 0 \text{ for discrete subset of } X)$$

For compact RS: finiteness of the support $\Rightarrow$ $\exists$ of degree f-ion

Def: The degree of a divisor D on X = cpct RS is
$$\deg(D) := \sum_{p \in X} D(p).$$

$\deg : \mathbb{Z}^X \equiv \operatorname{Div}(X) \to \mathbb{Z}$ is a group homomorphism,
$\operatorname{Ker}(\deg) = \operatorname{Div}_0(X) \subseteq \operatorname{Div}(X)$ (divisors of degree 0)

Def: (principal divisor) $f \in \mathcal{M}^*(X)$ a non-zero meromorphic function. The divisor of f, $\operatorname{div}(f)$, is defined by $\operatorname{div}(f) := \sum_p \operatorname{ord}_p(f) \cdot p$ and called a principal divisor on X. $\operatorname{PDiv}(X)$ ... princ. div. on X (a set)

We already know ($\Leftarrow$ properties of ord - f-ion)

1/ $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$,

2/ $\operatorname{div}\left(\frac{f}{g}\right) = \operatorname{div}(f) - \operatorname{div}(g)$,

3/ $\operatorname{div}\left(\frac{1}{f}\right) = -\operatorname{div}(f)$.

$\Rightarrow$ $\operatorname{PDiv}(X) \subseteq \operatorname{Div}(X)$ is a subgroup

Because if $f \in \mathcal{M}^*(X)$, $\deg(\operatorname{div}(f)) = 0$ (on compact RS), which is equivalent to $\sum_{p \in X} \operatorname{ord}_p(f) = 0$.

Ex: $X = \mathbb{P}^1/\mathbb{C} = \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, the Riemann sphere, z … coordinate

Any meromorph fction on $\mathbb{P}^1/\mathbb{C}$ is $f(z) = c \prod_{i=1}^{n} (z-\lambda_i)^{e_i}$

$\lambda_i \in \mathbb{C}$ mutually distinct,

$e_i \in \mathbb{Z}$

$$\mathrm{div}(f) = \sum_{i=1}^{n} e_i \cdot \lambda_i - \left(\sum_{i=1}^{n} e_i\right) \cdot \infty .$$

Ex: $\theta(z)$ … theta fction, holom + simple zeroes at $1/2 + \tau/2 + \ell$,

$\ell \in \mathbb{Z} + \mathbb{Z}\tau$

(lattice)

$$\mathrm{div}(\theta) = \sum_{m,n \in \mathbb{Z}}' 1 \cdot (1/2) + (\tau/2) + m + n\tau$$

(this divisor on $\mathbb{C}$ does not have finite support)

Def: The divisor of zeroes and poles of $f \in \mathcal{M}^*(X)$:

$$\mathrm{div}_0(f) := \sum_{\substack{p \in X \\ \mathrm{ord}_p(f) > 0}}' \mathrm{ord}_p(f) \cdot p ,$$

$$\mathrm{div}_\infty(f) := \sum_{\substack{p \in X \\ \mathrm{ord}_p(f) < 0}}' (-\mathrm{ord}_p(f)) \cdot p$$

$\mathrm{div}(f) = \underbrace{\mathrm{div}_0(f) - \mathrm{div}_\infty(f)}$

both positive divisors

(coef. are positive)

Now, ω is a non-zero meromorphic 1-form (not identically zero.)

Def: The divisor of ω is defined by

$$\mathrm{div}(\omega) = \sum_{p \in X}' \mathrm{ord}_p(\omega) \cdot p$$

and called canonical divisor on X. The set of can. div. is denoted $K\mathrm{Div}(X)$.

Ex: $\omega = dz$ on $\mathbb{P}^1/\mathbb{C}$; then $\mathrm{div}(\omega) = -2 \cdot \infty$, ω has no zeroes and the double pole at ∞. For $\omega = f(z)\,dz$, $f = c \prod_{i=1}^{n} (z-\lambda_i)^{e_i}$ a rational fction,

$$\mathrm{div}(\omega) = \sum_i e_i \cdot \lambda_i - \left(2 + \sum e_i\right) \cdot \infty .$$

All such forms have degree $= -2$.

$f \in \mathcal{M}^*(X)$, ω a meromorphic 1-form on X, then $\operatorname{div}(f\omega) = \operatorname{div}(f) + \operatorname{div}(\omega)$.

Lemma: ω_1, ω_2 ... meromorphic 1-forms on X, $\omega_1 \neq 0$ (identically). Then $\exists$ a unique meromorphic fcion $f \in \mathcal{M}^*(X)$ with $\omega_2 = f\omega_1$.

Pf: $\varphi: U \to V \subseteq \mathbb{C}$ a chart on X, complex coordinate z. Write $\omega_i = g_i(z)dz$ for $g_i(z) \in \mathcal{M}^*(X)|_V$, and define $f = h \circ \varphi$ a meromorphic fcion on U with $h = \frac{g_2}{g_1} \in \mathcal{M}^*(V)$. It is easy to verify that f is well-defined, independent of coordinate chart choice. $\square$

Corollary: $K\operatorname{Div}(X)$ is principal homogeneous space for $P\operatorname{Div}(X)$. (i.e., the difference of two divisors in $K\operatorname{Div}(X)$ is a divisor in $P\operatorname{Div}(X)$.) Therefore, $K\operatorname{Div}(X) = \operatorname{div}(\omega) + P\operatorname{Div}(X)$ for any non-zero meromorphic 1-form.

Let $f \in \mathcal{M}^*(X)$, regarded as holomorphic map $F: X \to \mathbb{P}^1/\mathbb{C} = \mathbb{C}_\infty$, $\deg(F) = d$, genus of X is g; by Hurwitz formula for X, $\mathbb{P}^1/\mathbb{C}$:

$$\sum_p (\operatorname{mult}_p(F) - 1) = 2g - 2 + 2\deg(F).$$

Consider merom. form $\omega = dz$ on $\mathbb{P}^1/\mathbb{C}$, $\deg(\omega) = -2$ (double pole at ∞, no other zeroes or poles,) $\eta = F^*(dz)$ its pull-back to X.

$$\boxed{\deg(\operatorname{div}(\eta))} = \sum_{p \in X} \operatorname{ord}_p(\eta) = \sum_{p \in X} \operatorname{ord}_p(F^*\omega) =$$

$$= \sum_{p \in X} \left[\left(1 + \operatorname{ord}_{F(p)}(\omega)\right) \operatorname{mult}_p(F) - 1 \right]$$

$$= \sum_{\substack{q \notin \infty \\ p \in F^{-1}(q)}} [\operatorname{mult}_p(F) - 1] + \sum_{p \in F^{-1}(\infty)} [-\operatorname{mult}_p(F) - 1]$$

$$= \sum_{p \in X} [\operatorname{mult}_p(F) - 1] - \sum_{p \in F^{-1}(\infty)} 2\operatorname{mult}_p(F)$$

$$= 2g - 2 + 2\deg(F) - 2\deg(F)$$

$$= \boxed{2g - 2}$$

(40) Let $F: X \to Y$ be a non-constant map of RS.

<u>Def</u>: Let $q \in Y$ be a point. The inverse image divisor of q, $F^*(q)$, is the divisor
$$F^*(q) = \sum_{p \in F^{-1}(q)} \mathrm{mult}_p(F) \cdot p$$

(the degree of $F^*(q)$ is for X, Y compact independent of q and equal to $\deg(F)$.)

For $D = \sum_{q \in Y} n_q \cdot q$ a divisor on Y, the pull-back to X, $F^*(D)$, is the divisor $F^*(D) = \sum_{q \in Y} n_q F^*(q)$. The divisors, regarded as functions, are
$$(F^*(D))(p) = \mathrm{mult}_p(F)\, D(F(p)).$$

The behavior of pull-back w.r. to operations on divisors is given by

<u>Lemma</u>: $F: X \to Y$ non-constant holomorphic map, X, Y - RS. Then

(1) The pull-back is a group homomorphism $F^*: \mathrm{Div}(Y) \to \mathrm{Div}(X)$,

(2) —— " —— of principal divisor is principal: if f is a merom. fction, $F^*(\mathrm{div}(f)) = \mathrm{div}(F^*(f)) = \mathrm{div}(f \circ F)$.

(3) X, Y - compact, so divisors have degree; then
$$\deg(F^*(D)) = \deg(F)\deg(D).$$

<u>Pf</u>: Straightforward, e.g. F^* extends by linearity from the pull-back of a point, and (1) follows. Analogously for (2), (3). □

<u>Def</u>: The ramification divisor of F, R_F, is the divisor on X defined by
$$R_F := \sum_{p \in X} [\mathrm{mult}_p(F) - 1] \cdot p$$

The branch divisor of F, B_F, is the divisor on Y defined by
$$B_F := \sum_{y \in Y} \Big[\sum_{p \in F^{-1}(y)} (\mathrm{mult}_p(F) - 1) \Big] \cdot y$$

The degree of R_F, B_F are equal, and ~~both~~ are also equal to the error term in the Hurwitz formula:

(41) $$2g(X) - 2 = \deg(F)\,(2g(Y) - 2) + \deg(R_F).$$

More precisely, we have

Lemma: $F: X \to Y$ holom. map between RS. Let ω be a non-zero mer. 1-form on Y. Then
$$\operatorname{div}(F^*\omega) = F^*(\operatorname{div}(\omega)) + R_F.$$

For X, Y compact, the application of deg to previous formula yields the Hurwitz formula.

For $D \in \operatorname{Div}(X)$, $D \geq 0$ if $D(p) \geq 0$ $\forall p \in X$.
We write $D > 0$ if $D \geq 0$ and $D \neq 0$. We write $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$ (similarly for $>$) $\Rightarrow$ there is partial ordering on the set $\operatorname{Div}(X)$. $\forall D \in \operatorname{Div}(X)$ can be (uniquely) written as
$$D = P - N, \quad P \geq 0, \quad N \geq 0$$
with disjoint support.

For $f \in \mathcal{M}^*(X)$, f is holomorphic iff $\operatorname{div}(f) \geq 0$.
If $f, g \in \mathcal{M}^*(X)$ such that $f + g \in \mathcal{M}^*(X)$, then
$$\operatorname{div}(f+g) \geq \min\{\operatorname{div}(f), \operatorname{div}(g)\},$$
since the same holds for orders.

<u>Def</u>: $D_1, D_2 \in \operatorname{Div}(X)$ on RS X are linearly equivalent, $D_1 \sim D_2$, if $D_1 - D_2 = (f)$, $f \in \mathcal{M}^*(X)$.

Linear equivalence is an equivalence relation on the set $\operatorname{Div}(X)$, a divisor is linearly equivalent to 0 iff it is a principal divisor and for X compact, linearly equivalent divisors have the same degree ($D_1 \sim D_2 \Rightarrow \deg D_1 = \deg D_2$).

 X - RS; if $f \in \mathcal{M}^*(X)$, then $\mathrm{div}_0(f) \sim \mathrm{div}_\infty(f)$.

Any two canonical divisors are (linearly) equivalent and $\forall$ divisor linearly equivalent to a canonical divisor is a canonical divisor.

If X is the Riemann sphere $\mathbb{P}^1/\mathbb{C}$, any two points on X are linearly equivalent. If $F: X \to Y$ is a holomorphic map, $D_1 \sim D_2$ divisors on Y, then $F^*(D_1) \sim F^*(D_2)$ on X.

Ex (Riemann sphere)

A divisor $D \in \mathrm{Div}(\mathbb{P}^1/\mathbb{C})$ is a principal divisor iff $\deg(D) = 0$.

Pf: Suppose $\deg(D) = 0$, $D = \sum_i e_i \cdot \lambda_i + e_\infty \cdot \infty$, $e_\infty = -\sum_i e_i$.

Then $D = \mathrm{div}(f)$, where $f(z) = \prod_i (z - \lambda_i)^{e_i}$. This is the sufficiency condition, the necessity is clear. $\square$

$D_1, D_2 \in \mathrm{Div}(\mathbb{P}^1/\mathbb{C})$, $D_1 \sim D_2$ ~~then~~ iff $\deg(D_1) = \deg(D_2)$.

Ex. (Complex torus)

$X = \mathbb{C}/L$ is an algebraic (commutative) group. We introduce

$A: \mathrm{Div}(X) \to X$; A is a group homomorphism, called

$\sum_i n_i \cdot p_i \mapsto \sum_i n_i p_i$ Abel - Jacobi map.

Abel's theorem for complex torus: A divisor $D \in \mathrm{Div}(\mathbb{C}/L)$ on X is principal iff $\deg(D) = 0$ & $A(D) = 0$.

group unit in X

It is now convenient to define $\mathrm{ord}_p(f) = \infty$ if $f \equiv 0$ in a neighb. of p. We also use the convention $\infty > n$ $\forall n \in \mathbb{Z}$.

Def: $L(D)$ = the space of merom. fcions with poles bounded by D, denoted $L(D)$:

$$L(D) = \{ f \in \mathcal{M}(X) \mid \mathrm{div}(f) \geq -D \}, \quad D \in \mathrm{Div}(X)$$

$\Rightarrow$ $L(D)$ is a vector space

Ex: Suppose $D(p) = n > 0$. Then if $f \in L(D)$, we must have $\mathrm{ord}_p(f) \geq -n$ (i.e., f may have a pole of order n at p, no worse.)

Or, for $D = \sum n_p p$, and local coordinate z_p around $p \in X$, the elements of $L(D)$ are fcions with Laurent series at p having no terms lower than $z_p^{-n_p}$ (for all p, otherwise holomorphic on $X \setminus D$.)

If $D_1 \leq D_2$, we have $L(D_1) \subseteq L(D_2)$.

A merom. fcion is holom. iff $\mathrm{div}(f) \geq 0$, thus

$$L(0) = \mathcal{O}(X) = \{\text{holom. fcions on } X\}.$$

If X is compact, $L(0) = \{\text{constant functions on } X\} \simeq \mathbb{C}$. (constant fcions)

<u>Lemma</u>: X - compact RS, $D \in \mathrm{Div}(X)$ with $\deg(D) < 0$.
Then $L(D) = \{0\}$.

<u>Pf</u>: Suppose $f \in L(D)$, $f \neq 0$. Take $D' := \mathrm{div}(f) + D$.
Since $f \in L(D)$, $E \geq 0$, and so $\deg(E) \geq 0$.
However, since $\deg(\mathrm{div}(f)) = 0$, we have
$\deg(E) = \underbrace{\deg(D) < 0}_{\text{by assumption}}$ $\Rightarrow$ contradiction,
there is no non-trivial $f \in L(D)$. $\square$

The complete linear system of D, is

$$|D| = \{E \in \mathrm{Div}(X) \mid E \sim D \ \& \ E \geq 0\}$$

We define

$$S : \mathbb{P}(L(D)) \to |D|$$

$$\langle f \rangle \mapsto \mathrm{div}(f) + D$$

(since $\mathrm{div}(\lambda f) = \mathrm{div}(f)$ $\forall \lambda \in \mathbb{C}$ $\Rightarrow$ S is well-defined.)

<u>Lemma</u>: X is compact RS, S is bijection.

<u>Pf</u>: Injectivity: $S(f) = S(g)$ as divisors. Then $\mathrm{div}(f) = \mathrm{div}(g)$, so $\mathrm{div}\left(\frac{f}{g}\right) = 0$ $\Rightarrow$ $\frac{f}{g}$ has no zeroes & poles on X. Since X is compact, $\frac{f}{g}$ = constant (non-zero) $\Rightarrow$ $\langle f \rangle = \langle g \rangle \in L(D)$.
Surjectivity is analogous. $\square$

Prop: Suppose D_1, D_2 ($\in \mathrm{Div}(X)$) are equivalent, $D_1 \sim D_2$, on X. Write $D_1 = D_2 + \mathrm{div}(h)$, $h \in \mathcal{M}^*(X)$. Then the mult. by h gives an isom. of $\mathbb{C}$-spaces

$$\mu_h : L(D_1) \xrightarrow{\sim} L(D_2).$$

Pf: Suppose $f \in L(D_1)$, $\mathrm{div}(f) \geq -D_1$. Then

$\mathrm{div}(hf) = \mathrm{div}(h) + \mathrm{div}(f) \geq \mathrm{div}(h) - D_1 = -D_2$

$\Rightarrow \quad hf = \mu_h(f) \in L(D_2)$. Thus

$\mu_h : L(D_1) \to L(D_2)$, $\mu_{h^{-1}} : L(D_2) \to L(D_1)$ are inverses each other $\Rightarrow \mu_h$ is an isomorphism. $\square$

Analogous considerations for 1-forms:

Def: The space of $\omega \in \mathcal{M}^{(1)}(X)$ with poles bounded by D, denoted by $L^{(1)}(D)$, is defined by

$$L^{(1)}(D) := \{\omega \in \mathcal{M}^{(1)}(X) \mid \mathrm{div}(\omega) \geq -D\}.$$

Again, $L^{(1)}(D)$ is a $\mathbb{C}$-vector space, $L^{(1)}(0) = \Omega^1(X)$ (holom. 1-forms on X).

If $D_1 \sim D_2$, given by $h \in \mathcal{M}^*(X)$, $\mu_h : L^{(1)}(D_1) \xrightarrow{\sim} L^{(1)}(D_2)$.

Fix $K = \mathrm{div}(\omega)$ a canonical divisor (for ω a meromophic 1-form) and $D \in \mathrm{Div}(X)$. Let $f \in L(D+K)$, i.e., $\mathrm{div}(f) + D + K \geq 0$. Consider meromorp. 1-form $f\omega$: $\mathrm{div}(f\omega) = \mathrm{div}(f) + \mathrm{div}(\omega) = \mathrm{div}(f) + K \Rightarrow \mathrm{div}(f\omega) + D \geq 0$, $f\omega \in L^{(1)}(D)$. Therefore the multiplication by ω yields a $\mathbb{C}$-linear map

$$\mu_\omega : L(D+K) \to L^{(1)}(D).$$

<u>Lemma:</u> μ_ω is an isomorphism of $\mathbb{C}$-vector spaces,

$$\mu_\omega : L(D+K) \xrightarrow{\sim} L^{(1)}(D).$$

Pf: <u>Surjectivity</u>: choose a 1-form $\omega' \in L^{(1)}(D)$, $\operatorname{div}(\omega') + D \geq 0$. We know $\exists$ $f \in \mathcal{M}^*(X)$: $\omega' = f\omega$. Then

$$\operatorname{div}(f) + D + K = \operatorname{div}(f) + D + \operatorname{div}(\omega) = \operatorname{div}(f\omega) + D =$$
$$= \operatorname{div}(\omega') + D \geq 0, \text{ so } f \in L(D+K).$$

Injectivity is analogous. $\square$

<u>Ex</u>: ($L(D)$ on $\mathbb{P}^1/\mathbb{C}$.) $D \in \operatorname{Div}(\mathbb{P}^1/\mathbb{C})$, $\deg(D) \geq 0$,

$D = \sum_{i=1}^{n} e_i \cdot \lambda_i + e_\infty \cdot \infty$, $\lambda_i \in \mathbb{C}$ distinct, $\sum_{i=1}^{n} e_i + e_\infty \geq 0$.

Define $f_D(z) := \prod_{i=1}^{n} (z - \lambda_i)^{-e_i}$

<u>Lemma</u>: $L(D) = \{ g(z) f_D(z) \mid g(z) \text{ is a polynomial of degree at most } \deg(D) \}$

Pf: Fix $g(z)$ a polyn. of degree d; $\operatorname{div}(g) \geq -d \cdot \infty$. Because

$$\operatorname{div}(f_D) = \sum_i -e_i \cdot \lambda_i + \left(\sum_i e_i\right) \cdot \infty,$$

and so

$$\operatorname{div}(g f_D) + D = \operatorname{div}(g) + \operatorname{div}(f_D) + D$$
$$\geq \left(\sum_i e_i + e_\infty - d \right) \cdot \infty = \underbrace{(\deg(D) - d)}_{\text{at least } 0} \cdot \infty$$

$\Leftarrow$ if $d \leq \deg(D)$

$g f_D \in L(D)$.

Analogously, one proves g is a polynomial of degree at most $\deg(D)$. $\square$

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Corollary: $D \in \mathrm{Div}(\mathbb{P}^1/\mathbb{C})$. Then

$$\dim(L(D)) = \begin{cases} 0 & \text{if } \deg(D) < 0, \text{ and} \\ 1 + \deg(D) & \text{if } \deg(D) \geq 0. \end{cases}$$

This all immediately leads to various criterions for embeddings of X into projective spaces, Riemann-Roch theorem $\left(\dim L(D) - \dim L(K-D) = 1 + \deg D - \underbrace{\dim L(K)}_{= g}\right)$ $\forall D \in \mathrm{Div}(X)$

and many other applications.