(5) PBW – theorem for $\mathbb{C}_q[U/K_S]$: $\left.\begin{array}{l} P(S) \\ P_+(S) \\ P_{++}(S) \end{array}\right\}$ K-span $\langle \omega_i \rangle_{i\in S}$ $\quad K = \begin{array}{l}\mathbb{Z}\\ \mathbb{Z}_+ \\ \mathbb{N}\end{array}$

the complement
$S^c := \Delta \setminus S$;

The quantized algebra A_S^{hol} of holomorphic polynomials on U/K_S is defined $A_S^{hol} := \bigoplus_{\lambda \in P_+(S^c)} B_\lambda \subseteq \mathbb{C}_q[U]$

(A_S^{hol} is a right $U_q(g)$- comodule subalgebra of $\mathbb{C}_q[U]$)
a formula is multiplicity - free decomposition into irr. $U_q(g)$-mod

anti holomorphic polynom. $\leftrightarrow$ right $U_q(g)$-mod structure

Lemma: The linear subspace $A_S^0 := m\left((A_S^{hol})^* \otimes A_S^{hol}\right) \subseteq \mathbb{C}_q[U]$
right $U_q(g)$-mod, $*$-subalgebra of $\mathbb{C}_q[U]$.

(A_S^0 ~ quantization of the algebra of complex valued polynomials on the real manifold U/K_S^0) $\quad K_S^0 \equiv$ semi-simple part of K_S

for $q \to 1$, A_S^0 for $\# S^c = 1$ can be interpreted as polynomial algebra on double spherical G-variety.

$A_S \subseteq A_S^0$, the subspace of $U_q(\mathfrak{h})$ - invariant elements of A_S^0.
making it a right $U_q(g)$-mod $*$- subalgebra of $\mathbb{C}_q[U]$

Lemma. $A_S^0 \subseteq \mathbb{C}_q[U/K_S]$, i.e. $A_S \subseteq \mathbb{C}_q[U/K_S]$.
Furthermore,

$$A_S = \left\langle (C^\lambda_{v, v_\lambda})^* (C^\lambda_{w, v_\lambda}) \mid \lambda \in P_+(S^c),\ v, w \in V(\lambda) \right\rangle$$

Theorem: The factorized $*$-subalgebra A_S is equal to $\mathbb{C}_q[U/K_S]$ if
1/ $S = \phi$, i.e. $U/K_S = U/T$ is the full flag manifold,
2/ $\# S^c = 1$ and the simple root $\alpha \in S^c$ is a Gelfand's node.

(6)

Well-known:

U ... conn., simply conn. compact Lie group, Lie $(U) = \mathfrak{u}$

$\mathfrak{p} \subseteq \mathfrak{g}$ maximal (standard) parab. subalgebra

$K \subseteq U$ connected subgrp., Lie $(K) = \mathfrak{k}$, $\mathfrak{k} = \mathfrak{p} \cap \mathfrak{u}$.

Then (U, K) is a Gelfand pair iff either

1/ (U, K) is comp. Herm. symm. pair, or

2/ $(U, K) \simeq (SO(2\ell+1), U(\ell))$, $\ell \geq 2$, or

3/ $(U, K) = (Sp(2\ell), U(1) \times Sp(2\ell-1))$, $\ell \geq 2$

Th: $\sigma \in W^S$. Then π_σ restricts to an irreducible $*$-repr. of the factorized $*$-algebra A_S, and in particular, π_σ restricts to an irred. $*$-repr. of $\mathbb{C}_q[U/K_S]$.

Proof: Based on analysis of the action of self-adjoint operators $\pi_\sigma((C^\lambda_{M_i;\lambda})^* C^\lambda_{M_i;\lambda})$ on the Hilbert space $L_2(\mathbb{Z}_+) \otimes \ell(\sigma)$

[call it the representation of $\mathbb{C}_q[U/K_S]$ associated to the Schubert cell $X_\sigma \subseteq U/K_S$, $\sigma \in W^S$]

Th. $\sigma \in W$, $\sigma = u \cdot v$ a unique decomposition of $\sigma \in W$, $u \in W^S$, $v \in W_S$.

For $\pi_\sigma = \pi_u \otimes \pi_v$, $t \in T$, we have

$$(\pi_\sigma \otimes \tau_t)(a) = \pi_u(a) \otimes Id \otimes \ell(v), \quad a \in \mathbb{C}_q[U/K_S].$$

The $*$-representations $\{\pi_\sigma\}_{\sigma \in W^S}$ as $*$-repr. of A_S are mutually inequivalent.

As for the completeness, one has to consider completions of $\mathbb{C}_q[U]$ resp. $\mathbb{C}_q[U/K_S]$ w.r. to universal C^*-norm.

⑦

denoted $\mathbb{C}_q(U)$ resp. $\mathbb{C}_q(U/K_s)$:

$$\|a\|_u := \sup_{\sigma \in W, t \in T} \|(\pi_\sigma \otimes \tau_t)(a)\|$$

$a \in \mathbb{C}_q[U]$